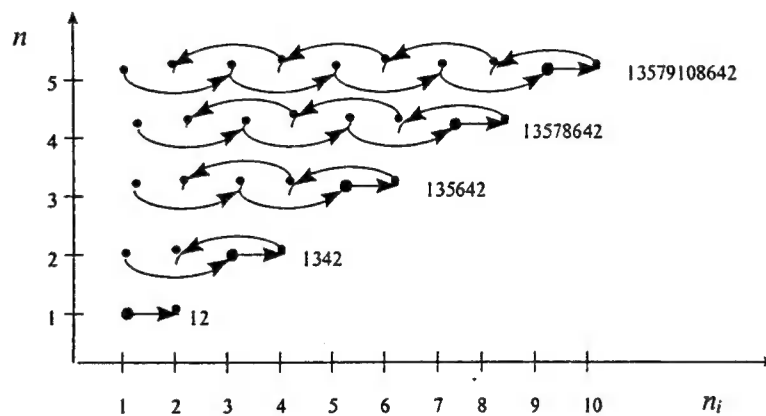


# PROCEEDINGS OF THE FIRST INTERNATIONAL CONFERENCE ON SMARANDACHE TYPE NOTIONS IN NUMBER THEORY

second edition



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PROCEEDINGS OF THE FIRST INTERNATIONAL CONFERENCE  
ON  
SMARANDACHE TYPE NOTIONS IN NUMBER THEORY

papers selected and edited

by

C.Dumitrescu & V.Seleacu

Number Theory Association of the University of Craiova

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## Summary of Conference Arrangements

The *First International Conference on Smarandache Notions in Number Theory* was held in Craiova, Romania, 21- 22 August 1997. The Organizing Committee had spared no effort in preparing programme, lodging and conference facilities. The Conference was opened by the late professor Constantin Dumitrescu<sup>1</sup>, chairman of the Organizing Committee and the initiator of the conference and a leading personality in Number Theory research. He welcomed all participants. Unfortunately professor Dumitrescu's state of health did not permit him to actively lead the conference, although he delivered his first paper later in the day and was present during most sessions. He requested the author of these lines to chair the first day of the conference, a task for which I was elected to continue for the rest of the conference.

In view of the above it is appropriate that I express mine and the other participants gratitude to the organizers and in particular to the Dumitrescu family who assisted throughout with social and arrangements and the facilities required for the smooth running of the conference. I would like to pay special tribute to professor Dumitrescu's son Antoniu Dumitrescu who presented his father's second paper on his behalf.

Unfortunately not all those who intended to participate in the conference were able to come. Their contributions which were submitted in advance have been gratefully received and are included in these proceedings. A list of participants is found on page .....

A pre-conference session was held with professor V. Seleacu the day before the conference. This was held in french with Mrs Dumitrescu as interpreter. Prof. Seleacu showed some interesting work being conducted by the research group at Craiova University. Mrs Dumitrescu also acted actively during the conference to bridge language difficulties.

Special thanks were expressed at the conference to Dr. F. Luca, USA, who helped during sessions when translation from the romanian language to english was needed. In this context thanks are also due to my wife Anne-Marie Rochard-Ibstedt who made my participation possible by helping me drive from Sweden to Paris and then across Europe to Craiova. She was also active during the conference in taking photos and distributing documents.

Although united through the international language of Mathematics it was not always possible to penetrate presentations in such detail that extended discussions could take place after each session. Informal contacts between participants proved important and opportunities for this was given during breaks and joint dinners.

In the concluding remarks the chairman thanked the organizers and in particular professor Dumitrescu for having very successfully organized this conference. It was noted that the presentations were not made as an end in itself but as sources for further thought and research in this particular area of Number Theory, n.b. the very large number of open problems and notions formulated by Florentin Smarandache. The hope was expressed that the conference had linked together researchers for continuing exchange of views with our modern means of communication such as electronic mail and high speed personal computers.

Professor Dumitrescu thanked the chairman for his work.

Paris 26 March 1998.

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<sup>1</sup> 1949-1997, Obituary in Vol. 8 of the Smarandache Notions Journal.



## On Smarandache's Periodic Sequences

Henry Ibstedt

### Preamble:

Ladies and gentlemen,

It is for me a great honour and a great pleasure to be here at this conference to present some of the thoughts I have given to a few of the ideas and research suggestions given by Florentin Smarandache. In both of my presentations we will look at some integer sequences defined by Smarandache. As part of my work on this I have prepared an inventory of Smarandache sequences, which is probably not complete, but nevertheless it contains 133 sequences. I welcome contributions to complete this inventory, in which an attempt is also made to classify the sequences according to certain main types.

Before giving my first presentation I would like to say a few words about what eventually brought me here.

When I was young my interest in Mathematics began when I saw the beauty of Euclidean geometry - the rigor of a mathematical structure built on a few axioms which seemed the only ones that could exist. That was long before I heard of the Russian mathematician Lobachevsky and hyperbolic geometry. But my fascination for Mathematics and numbers was awoken and who can dispute the incredible beauty of a formula like

$$e^{ix} + 1 = 0$$

and many others. But there was also the disturbing fact that many important truths can not be expressed in closed formulas and that more often than not we have to resort to approximations and descriptions. For a long time I was fascinated by classical mechanics. Newton's laws provided an ideal framework for a great number of interesting problems. But Einstein's theory of relativity and Heisenberg's uncertainty relation put a stop to living and thinking in such a narrow world. Eventually I ended up doing computer applications in Atomic Physics. But also my geographical world became too narrow and I started working in developing countries in Africa, the far East and the Caribbean, far away from computers, libraries and contact with current research. This is when I returned to numbers and Number Theory. In 1979, when micro computers had just started making an impact, I bought one and brought it with me to the depths of Africa. Since then Computer Analysis in Number Theory has remained my major intellectual interest and stimulant.

With these words I would now like to proceed to the subject of this session.

# On Smarandache's Periodic Sequences

Henry Ibstedt

## Abstract:

This paper is based on an article in Mathematical Spectrum, Vol. 29, No 1. It concerns what happens when an operation applied to an n-digit integer results in an n digit integer. Since the number of n-digit integers is finite a repetition must occur after applying the operation a finite number of times. It was assumed in the above article that this would lead to a periodic sequence which is not always true because the process may lead to an invariant. The second problem with the initial article is that, say, 7 is considered as 07 or 007 as the case may be in order make its reverse to be 70 or 700. However, the reverse of 7 is 7. In order not to loose the beauty of these sequences the author has introduced stringent definitions to prevent the sequences from collapse when the reversal process is carried out.

Four different operations on n-digit integers is considered.

The Smarandache n-digit periodic sequence. Definition: Let  $N_k$  be an integer of at most n digits and let  $R_k$  be its reverse.  $N'_k$  is defined through

$$N'_k = R_k \cdot 10^{n-1-\lfloor \log_{10} N_k \rfloor}$$

The element  $N_{k+1}$  of the sequence through

$$N_{k+1} = |N_k - N'_k|$$

where the sequence is initiated by an arbitrary n-digit integer  $N_1$  in the domain  $10^n \leq N_1 < 10^{n+1}$ .

The Smarandache Subtraction Periodic Sequence: Definition: Let  $N_k$  be a positive integer of at most n digits and let  $R_k$  be its digital reverse.  $N'_k$  is defined through

$$N'_k = R_k \cdot 10^{n-1-\lfloor \log_{10} N_k \rfloor}$$

The element  $N_{k+1}$  of the sequence through

$$N_{k+1} = |N'_k - c|$$

where c is a positive integer. The sequence is initiated by an arbitrary positive n-digit integer  $N_1$ . It is obvious from the definition that  $0 \leq N_k < 10^{n+1}$ , which is the range of the iterating function.

The Smarandache Multiplication Periodic Sequence: Definition: Let  $c > 1$  be a fixed integer and  $N_0$  and arbitrary positive integer.  $N_{k+1}$  is derived from  $N_k$  by multiplying each digit x of  $N_k$  by c retaining only the last digit of the product cx to become the corresponding digit of  $N_{k+1}$ .

The Smarandache Mixed Composition Periodic Sequence: Definition. Let  $N_0$  be a two-digit integer  $a_1 \cdot 10 + a_0$ . If  $a_1 + a_0 < 10$  then  $b_1 = a_1 + a_0$  otherwise  $b_1 = a_1 + a_0 + 1$ .  $b_0 = |a_1 - a_0|$ . We define  $N_1 = b_1 \cdot 10 + b_0$ .  $N_{k+1}$  is derived from  $N_k$  in the same way

Starting points for loops (periodic sequences), loop length and the number of loops of each kind has been calculated and displayed in tabular form in all four cases. The occurrence of invariants has also been included.

## Introduction

In *Mathematical Spectrum*, vol 29 No 1 [1], is an article on Smarandache's periodic sequences which terminates with the statement:

*"There will always be a periodic sequence whenever we have a function  $f:S \rightarrow S$ , where  $S$  is a finite set of positive integers and we repeat the function  $f$ ."*

We must adjust the above statement by a counterexample before we look at this interesting set of sequences. Consider the following trivial function  $f(x_k):S \rightarrow S$ , where  $S$  is an ascending set of integers  $\{a_1, a_2, \dots, a_r, \dots, a_n\}$ :

$$f(x_k) = \begin{cases} x_{k-1} & \text{if } x_k > a_r \\ x_k & \text{if } x_k = a_r \\ x_{k+1} & \text{if } x_k < a_r \end{cases}$$

As we can see the iteration of the function  $f$  in this case converges to an invariant  $a_r$ , which we may of course consider as a sequence (or loop) of only one member. We will however make a distinction between a sequence and an invariant in this paper.

There is one more snag to overcome. In the Smarandache sequences 05 is considered as a two-digit integer. The consequence of this is that 00056 is considered as a five digit integer while 056 is considered as a three-digit integer. We will abolish this ambiguity, 05 is a one-digit integer and 00200 is a three-digit integer.

With these two remarks in mind let's look at these sequences. There are in all four different ones reported in the above mentioned article in *Mathematical Spectrum*. The study of the first one will be carried out in much detail in view of the above remarks.

### 1a. The Two-Digit Smarandache Periodic Sequence

It has been assumed that the definition given below leads to a repetition according to Dirichlet's box principle (or the statement made above). However, as we will see, this definition leads to a collapse of the sequence.

**Preliminary definition.** Let  $N_k$  be an integer of at most two digits and let  $N_k'$  be its digital reverse. We define the element  $N_{k+1}$  of the sequence through

$$N_{k+1} = |N_k - N_k'|$$

where the sequence is initiated by an arbitrary two digit integer  $N_1$ .

Let's write  $N_1$  in the form  $N_1 = 10a + b$  where  $a$  and  $b$  are digits. We then have

$$N_2 = |10a + b - 10b - a| = 9 \cdot |a - b|$$

The  $|a - b|$  can only assume 10 different values 0, 1, 2, ..., 9. This means that  $N_3$  is generated from only 10 different values of  $N_2$ . Let's first find out which two digit integers result in  $|a - b| = 0, 1, 2, \dots$  and 9 respectively.

|a-b| Corresponding two digit integers

0	11	22	33	44	55	66	77	88	99								
1	10	12	21	23	32	34	43	45	54	56	65	67	76	78	87	89	98
2	13	20	24	31	35	42	46	53	57	64	68	75	79	86	97		
3	14	25	30	36	41	47	52	58	63	69	74	85	96				
4	15	26	37	40	48	51	59	62	73	84	95						
5	16	27	38	49	50	61	72	83	94								
6	17	28	39	60	71	82	93										
7	19	29	70	81	92												
8	19	80	91														
9	90																

It is now easy to follow the iteration of the sequence which invariably terminates in 0, table 1.

Table 1. Iteration of sequence according to the preliminary definition

a-b	N <sub>2</sub>	N <sub>3</sub>	N <sub>4</sub>	N <sub>5</sub>	N <sub>5</sub>	N <sub>6</sub>
0	0					
1	9	0				
2	18	63	27	45	9	0
3	27	45	9	0		
4	36	27	45	9	0	
5	45	9	0			
6	54	9	0			
7	63	27	45	9	0	
8	72	45	9	0		
9	81	63	27	45	9	0

The termination of the sequence is preceded by the one digit element 9 whose reverse is 9. The following definition is therefore proposed.

**Definition of Smarandache's two-digit periodic sequence.** Let  $N_k$  be an integer of at most two digits.  $N_k'$  is defined through

$$N_k' = \begin{cases} \text{the reverse of } N_k \text{ if } N_k \text{ is a two digit integer} \\ N_k \cdot 10 \text{ if } N_k \text{ is a one digit integer} \end{cases}$$

We define the element  $N_{k+1}$  of the sequence through

$$N_{k+1} = |N_k - N_k'|$$

where the sequence is initiated by an arbitrary two digit integer  $N_1$  with unequal digits.

Modifying table 1 according to the above definition results in table 2.

Table 2. Iteration of the Smarandache two digit sequence

a-b	N <sub>2</sub>	N <sub>3</sub>	N <sub>4</sub>	N <sub>5</sub>	N <sub>5</sub>	N <sub>6</sub>	N <sub>7</sub>
1	9	81	63	27	45	9	
2	18	63	27	45	9	81	63
3	27	45	9	81	63	27	
4	36	27	45	9	81	63	27
5	45	9	81	63	27	45	
6	54	9	81	63	27	45	9
7	63	27	45	9	81	63	
8	72	45	9	81	63	27	45
9	81	63	27	45	9	81	

**Conclusion:** The iteration always produces a loop of length 5 which starts on the second or the third term of the sequence. The period is 9, 81, 63, 27, 45 or a cyclic permutation thereof.

#### 1b. Smarandache's n-digit periodic sequence.

Let's extend the definition of the two-digit periodic sequence in the following way.

#### Definition of Smarandache's n-digit periodic sequence.

Let  $N_k$  be an integer of at most n digits and let  $R_k$  be its reverse.  $N_k'$  is defined through

$$N_k' = R_k \cdot 10^{n-1-\lfloor \log_{10} N_k \rfloor}$$

We define the element  $N_{k+1}$  of the sequence through

$$N_{k+1} = |N_k - N_k'|$$

where the sequence is initiated by an arbitrary n-digit integer  $N_1$  in the domain  $10^n \leq N_1 < 10^{n+1}$ . It is obvious from the definition that  $0 \leq N_k < 10^{n+1}$ , which is the range of the iterating function.

Let's consider the cases  $n=3$ ,  $n=4$ ,  $n=5$  and  $n=6$ .

$n=3$ .

Domain  $100 \leq N_1 \leq 999$ . The iteration will lead to an invariant or a loop (periodic sequence)<sup>1</sup>. There are 90 symmetric integers in the domain, 101, 111, 121, ..., 202, 212, ..., for which  $N_2=0$  (invariant). All other initial integers iterate into various entry points of the same periodic sequence. The number of numbers in the domain resulting in each entry of the loop is denoted s in table 3.

Table 3. Smarandache 3-digit periodic sequence

s	239	11	200	240	120
Loop	99	891	693	297	495

It is easy to explain the relation between this loop and the loop found for  $n=2$ . Consider  $N=a_0+10a_1+100a_2$ . From this we have  $|N-N'|=99|a_2-a_0|=11 \cdot 9|a_2-a_0|$  which is 11 times the corresponding expression for  $n=2$  and as we can see this produces a 9 as middle (or first) digit in the sequence for  $n=3$ .

$n=4$ .

Domain  $1000 \leq N_1 \leq 9999$ . The largest number of iterations carried out in order to reach the first member of the loop is 18 and it happened for  $N_1=1019$ . The iteration process ended up in the invariant 0 for 182 values of  $N_1$ , 90 of these are simply the symmetric integers in the domain like  $N_1=4334$ , 1881, 7777, etc., the other 92 are due to symmetric integers obtained after a couple of iterations. Iterations of the other 8818 integers in the domain result in one of the following 4 loops or a cyclic permutation of one of these. The number of numbers in the domain resulting in each entry of the loops is denoted s in table 4.

<sup>1</sup> This is elaborated in detail in *Surfing on the Ocean of Numbers* by the author, Vail Univ. Press 1997.

Table 4. Smarandache 4-digit periodic sequences

s	378	259			
Loop	2178	6534			
s	324	18	288	2430	310
Loop	90	810	630	270	450
s	446	2	449	333	208
Loop	909	8181	6363	2727	4545
s	329	11	290	2432	311
Loop	999	8991	6993	2997	4995

**n=5.**

Domain  $10000 \leq N_1 \leq 99999$ . There are 900 symmetric integers in the domain. 920 integers in the domain iterate into the invariant 0 due to symmetries.

Table 5. Smarandache 5-digit periodic sequences

s	3780	2590			
Loop	21978	65934			
s	3240	180	2880	24300	3100
Loop	990	8910	6930	2970	4950
s	4469	11	4490	3330	2080
Loop	9009	81081	63063	27027	45045
s	3299	101	2900	24320	3110
Loop	9999	89991	69993	29997	49995

**n=6.**

Domain  $100000 \leq N_1 \leq 999999$ . There are 900 symmetric integers in the domain. 12767 integers in the domain iterate into the invariant 0 due to symmetries. The longest sequence of iterations before arriving at the first loop member is 53 for  $N=100720$ . The last column in table 6 shows the number of integers iterating into each loop.

Table 6. Smarandache 6-digit periodic sequences

[illegible]

## 2. The Smarandache Subtraction Periodic Sequence

**Definition:**

Let  $N_k$  be a positive integer of at most  $n$  digits and let  $R_k$  be its digital reverse.  $N_k'$  is defined through

$$N'_k = R_k \cdot 10^{n-1-\lfloor \log_{10} N_k \rfloor}$$

We define the element  $N_{k+1}$  of the sequence through

$$N_{k+1} = |N'_k - c|$$

where  $c$  is a positive integer. The sequence is initiated by an arbitrary positive  $n$ -digit integer  $N_1$ . It is obvious from the definition that  $0 \leq N_k < 10^{n+1}$ , which is the range of the iterating function.

$c=1, n=2, 10 \leq N_1 \leq 99$

When  $N_1$  is of the form  $11 \cdot k$  or  $11 \cdot k-1$  then the iteration process results in 0, see figure 1a.

Every other member of the interval  $10 \leq N_1 \leq 99$  is a entry point into one of five different cyclic periodic sequences. Four of these are of length 18 and one of length 9 as shown in table 7 and illustrated in figures 1b and 1c, where important features of the iteration chains are shown.

Table 7. The subtraction periodic sequence,  $10 \leq N_1 \leq 99$

Seq. No 1	12	20	1	9	89	97	78	86	67	75	56	64	45	53	34	42	23	31
Seq. No 2	13	30	2	19	90	■	79	96	68	85	57	74	46	63	35	52	24	41
Seq. No 3	14	40	3	29	91	18	80	7	69	95	58	84	47	73	35	62	25	51
Seq. No 4	15	50	4	39	92	28	81	17	70	6	59	94	48	83	37	72	26	61
Seq. No 5	16	60	5	49	93	38	82	27	71									

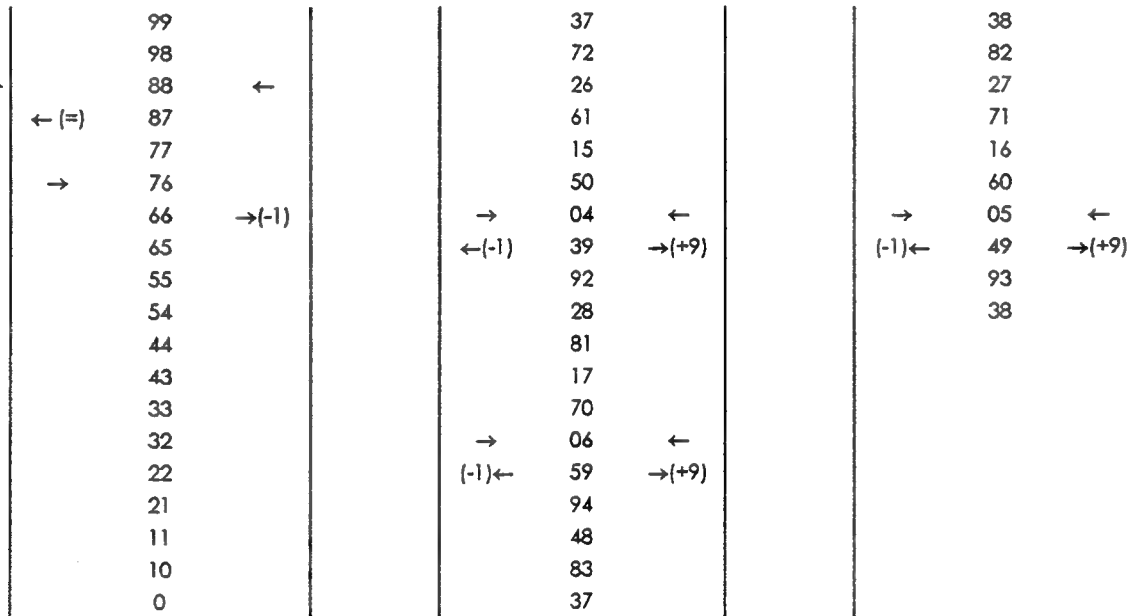


Fig. 1a

Fig 1b

Fig 1c

$1 \leq c \leq 9, n=2, 100 \leq N_1 \leq 999$

A computer analysis revealed a number of interesting facts concerning the application of the iterative function.

There are no periodic sequences for  $c=1, c=2$  and  $c=5$ . All iterations result in the invariant 0 after, sometimes, a large number of iterations.

For the other values of  $c$  there are always some values of  $N_1$  which do not produce periodic sequences but terminate on 0 instead. Those values of  $N_1$  which produce periodic sequences will either have  $N_1$  as the first term of the sequence or one of the values  $f$  determined by  $1 \leq f \leq c-1$  as first term. There are only eight different possible value for the length of the loops, namely 11, 22, 33, 50, 100, 167, 189, 200. Table 8 shows how many of the 900 initiating integers in the interval  $100 \leq N_1 \leq 999$  result in each type of loop or invariant 0 for each value of  $c$ .

Table 8. Loop statistics,  $L$ =length of loop,  $f$ =first term of loop

$c$	$f \downarrow / L \rightarrow$	0	11	22	33	50	100	167	189	200
1	$N_1$	900								
2	$N_1$	900								
3	$N_1$	241			59			150		
	1							240		
	2							210		
4	$N_1$	494				42				
	1					364				
5	$N_1$	900								
6	$N_1$	300			59		84			
	1						288			
	2						169			
7	$N_1$	109								535
	1									101
	2									101
	3									14
	4									14
	5									13
	6									13
8	$N_1$	203				43	85			
	1						252			
	2					305				
	3						12			
9	$N_1$	21	79	237					170	
	4								20	
	5								10	
	6		161							
	7			121						
	8			81						

A few examples:

For  $c=2$  and  $N_1=202$  the sequence ends in the invariant 0 after only 2 iterations:

202 200 0

For  $c=9$  and  $N_1=208$  a loop is closed after only 11 iterations:

208 793 388 874 469 955 550 46 631 127 712 208

For  $c=7$  and  $N_1=109$  we have an example of the longest loop obtained. It has 200 elements and the loop is closed after 286 iterations:

109 894 491 187 774 470 67 753 350 46 633 329 916 612 209 895 591 188 874 471  
167 754 450 47 733 330 26 613 309 896 691 189 974 472 267 755 550 48 833 331  
126 614 409 897 791 190 84 473 367 756 650 49 933 332 226 615 509 898 891 191  
184 474 467 757 750 50 43 333 326 616 609 899 991 192 284 475 567 758 850 51  
143 334 426 617 709 900 2 193 384 476 667 759 950 52 243 335 526 618 809 901  
102 194 484 477 767 760 60 53 343 336 626 619 909 902 202 195 584 478 867 761  
160 54 443 337 726 620 19 903 302 196 684 479 967 762 260 55 543 338 826 621  
119 904 402 197 784 480 77 763 360 56 643 339 926 622 219 905 502 198 884 481  
177 764 460 57 743 340 36 623 319 906 602 199 984 482 277 765 560 58 843 341  
136 624 419 907 702 200 5 493 387 776 670 69 953 352 246 635 529 918 812 211  
105 494 487 777 770 70 63 353 346 636 629 919 912 212 205 495 587 778 870 71  
163 354 446 637 729 920 22 213 305 496 687 779 970 72 263 355 546 638 829 921  
122 214 405 497 787 780 80 73 363 356 646 639 929 922 222 215 505 498 887 781  
180 74 463 357 746 640 39 923 322 216 605 499 987 782 280 75 563 358 846 641  
139 924 422 217 705 500 2



### 3. The Smarandache Multiplication Periodic Sequence

#### Definition:

Let  $c > 1$  be a fixed integer and  $N_0$  and arbitrary positive integer.  $N_{k+1}$  is derived from  $N_k$  by multiplying each digit  $x$  of  $N_k$  by  $c$  retaining only the last digit of the product  $cx$  to become the corresponding digit of  $N_{k+1}$ .

In this case each digit position goes through a separate development without interference with the surrounding digits. Let's as an example consider the third digit of a 6-digit integer for  $c=3$ . The iteration of the third digit follows the schema:

xx7yyy ---- the third digit has been arbitrarily chosen to be 7.  
 xx1yyy  
 xx3yyy  
 xx9yyy  
 xx7yyy ---- which closes the loop for the third digit.

Let's now consider all the digits of a six-digit integer 237456:

237456  
 691258  
 873654  
 419852  
 237456 ---- which closes the loop.

The digits 5 and 0 are invariant under this iteration. All other digits have a period of 4 for  $c=3$ .

**Conclusion:** Integers whose digits are all equal to 5 are invariant under the given operation. All other integers iterate into a loop of length 4.

We have seen that the iteration process for each digit for a given value of  $c$  completely determines the iteration process for any  $n$ -digit integer. It is therefore of interest to see these single digit iteration sequences:

Table 9. One-digit multiplication sequences

c=2	c=3	c=4	c=5
1 2 4 8 6 2	1 3 9 7 1	1 4 6 4	1 5 5
2 4 8 6 2	2 6 8 4 2	2 8 2	2 0 0
3 6 2 4 8 6	3 9 7 1 3	3 2 8 2	3 5 5
4 8 6 2 4	4 2 6 8 4	4 6 4	4 0 0
5 0 0	5 5	5 0 0	5 5
6 2 4 8 6	6 8 4 2 6	6 4 6	6 0 0
7 4 8 6 2 4	7 1 3 9 7	7 8 2 8	7 5 5
8 6 2 4 8	8 4 2 6 8	8 2 8	8 0 0
9 8 6 2 4 8	9 7 1 3 9	9 6 4 6	9 5 5

c=6	c=7	c=8	c=9
1 6 6	1 7 9 3 1	1 8 4 2 6 8	1 9 1
2 2	2 4 8 6 2	2 6 8 4 2	2 8 2
3 8 8	3 1 7 9 3	3 4 2 6 8 4	3 7 3
4 4	4 8 6 2 4	4 2 6 8 4	4 6 4
5 0 0	5 5	5 0 0	5 5
6 6	6 2 4 8 6	6 8 4 2 6	6 4 6
7 2 2	7 9 3 1 7	7 6 8 4 2 6	7 3 7
8 8	8 6 2 4 8	8 4 2 6 8	8 2 8
9 4 4	9 3 1 7 9	9 2 6 8 4 2	9 1 9

With the help of table 9 it is now easy to characterize the iteration process for each value of  $c$ .

Integers composed of the digit 5 result in an invariant after one iteration. Apart from this we have for:

$c=2$ . Four term loops starting on the first or second term.

$c=3$ . Four term loops starting with the first term.

$c=4$ . Two term loops starting on the first or second term (could be called a switch or pendulum).

$c=5$ . Invariant after one iteration.

$c=6$ . Invariant after one iteration.

$c=7$ . Four term loop starting with the first term.

$c=8$ . Four term loop starting with the second term.

$c=9$ . Two term loops starting with the first term (pendulum).

#### 4. The Smarandache Mixed Composition Periodic Sequence

**Definition.** Let  $N_0$  be a two-digit integer  $a_1 \cdot 10 + a_0$ . If  $a_1 + a_0 < 10$  then  $b_1 = a_1 + a_0$  otherwise  $b_1 = a_1 + a_0 + 1$ .  $b_0 = |a_1 - a_0|$ . We define  $N_1 = b_1 \cdot 10 + b_0$ .  $N_{k+1}$  is derived from  $N_k$  in the same way.<sup>2</sup>

There are no invariants in this case. 36, 90, 93 and 99 produce two-element loops. The longest loops have 18 elements. A complete list of these periodic sequences is presented below.

```

10 11 20 22 40 44 80 88 70 77 50 55 10
11 20 22 40 44 80 88 70 77 50 55 10 11
12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12
13 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42
14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14
15 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64
16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16
17 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86
18 97 72 95 54 91 18
19 18 97 72 95 54 91 18
20 22 40 44 80 88 70 77 50 55 10 11 20
21 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31
22 40 44 80 88 70 77 50 55 10 11 20 22
23 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51
24 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62
25 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73
26 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84
27 95 54 91 18 97 72 95
28 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16
29 27 95 54 91 18 97 72 95
30 33 60 66 30
31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31
32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32
33 60 66 30 33
34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34
35 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82
36 93 36
37 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14
38 25 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73
39 36 93 36

```

<sup>2</sup> Formulation conveyed to the author: "Let  $N$  be a two-digit number. Add the digits, and add them again if the sum is greater than 10. Also take the absolute value of their difference. These are the first and second digits of  $N_1$ ."

40 44 80 88 70 77 50 55 10 11 20 22 40  
41 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53  
42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42  
43 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71  
44 80 88 70 77 50 55 10 11 20 22 40 44  
45 91 18 97 72 95 54 91  
46 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12  
47 23 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51  
48 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34  
49 45 91 18 97 72 95 54 91  
50 55 10 11 20 22 40 44 80 88 70 77 50  
51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51  
52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52  
53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53  
54 91 18 97 72 95 54  
55 10 11 20 22 40 44 80 88 70 77 50 55  
56 21 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31  
57 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32  
58 43 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71  
59 54 91 18 97 72 95 54  
60 66 30 33 60  
61 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75  
62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62  
63 93 36 93  
64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64  
65 21 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31  
66 30 33 60 66  
67 41 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53  
68 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52  
69 63 93 36 93  
70 77 50 55 10 11 20 22 40 44 80 88 70  
71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71  
72 95 54 91 18 97 72  
73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73  
74 23 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51  
75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75  
76 41 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53  
77 50 55 10 11 20 22 40 44 80 88 70 77  
78 61 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75  
79 72 95 54 91 18 97 72  
80 88 70 77 50 55 10 11 20 22 40 44 80  
81 97 72 95 54 91 18 97  
82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82  
83 25 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73  
84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84  
85 43 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71  
86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86  
87 61 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75  
88 70 77 50 55 10 11 20 22 40 44 80 88  
89 81 97 72 95 54 91 18 97  
90 99 90  
91 18 97 72 95 54 91  
92 27 95 54 91 18 97 72 95  
93 36 93  
94 45 91 18 97 72 95 54 91  
95 54 91 18 97 72 95  
96 63 93 36 93  
97 72 95 54 91 18 97  
98 81 97 72 95 54 91 18 97  
99 90 99

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# Products of Factorials in Smarandache Type Expressions

Florian Luca

## Introduction

In [3] and [5] the authors ask how many primes are of the form  $x^y + y^x$ , where  $\gcd(x, y) = 1$  and  $x, y \geq 2$ . Moreover, Jose Castillo (see [2]) asks how many primes are of the Smarandache form  $x_1^{x_2} + x_2^{x_3} + \dots + x_n^{x_1}$ , where  $n > 1$ ,  $x_1, x_2, \dots, x_n > 1$  and  $\gcd(x_1, x_2, \dots, x_n) = 1$  (see [9]).

In this article we announce a lower bound for the size of the largest prime divisor of an expression of the type  $ax^y + by^x$ , where  $ab \neq 0$ ,  $x, y \geq 2$  and  $\gcd(x, y) = 1$ .

For any finite extension  $F$  of  $\mathbb{Q}$  let  $d_F = [F : \mathbb{Q}]$ . For any algebraic number  $\zeta \in F$  let  $N_F(\zeta)$  denote the norm of  $\zeta$ .

For any rational integer  $n$  let  $P(n)$  be the largest prime number  $P$  dividing  $n$  with the convention that  $P(0) = P(\pm 1) = 1$ .

**Theorem 1.** *Let  $\alpha$  and  $\beta$  be algebraic integers with  $\alpha \cdot \beta \neq 0$ . Let  $K = \mathbb{Q}[\alpha, \beta]$ . For any two positive integers  $x$  and  $y$  let  $X = \max(x, y)$ . There exist computable positive numbers  $C_1$  and  $C_2$  depending only on  $\alpha$  and  $\beta$  such that*

$$P(N_K(\alpha x^y + \beta y^x)) > C_1 \left( \frac{X}{\log^3 X} \right)^{1/(d_K+1)}$$

whenever  $x, y \geq 2$ ,  $\gcd(x, y) = 1$ , and  $X > C_2$ .

The proof of Theorem 1 uses lower bounds for linear forms in logarithms of algebraic numbers (see [1] and [7]) as well as an idea of Stewart (see [10]).

Erdős and Obláth (see [4]) found all the solutions of the equation  $n! = x^p \pm y^p$  with  $\gcd(x, y) = 1$  and  $p > 2$ . Moreover, the author (see [6]) showed that in every non-degenerate binary recurrence sequence  $(u_n)_{n \geq 0}$  there are only finitely many terms which are products of factorials.

We use Theorem 1 to show that for any two given integers  $a$  and  $b$  with  $ab \neq 0$ , there exist only finitely many numbers of the type  $ax^y + by^x$ , where  $x, y \geq 2$  and  $\gcd(x, y) = 1$ , which are products of factorials.

Let  $\mathcal{PF}$  be the set of all positive integers which can be written as products of factorials; that is

$$\mathcal{PF} = \{w \mid w = \prod_{j=1}^k m_j!, \text{ for some } m_j \geq 1\}.$$

**Theorem 2.** Let  $f_1, \dots, f_s \in \mathbb{Z}[X, Y]$  be  $s \geq 1$  homogeneous polynomials of positive degrees. Assume that  $f_i(0, Y) \cdot f_i(X, 0) \neq 0$  for  $i = 1, \dots, s$ . Then, the equation

$$f_1(x_1^{y_1}, y_1^{x_1}) \cdot \dots \cdot f_s(x_s^{y_s}, y_s^{x_s}) \in \mathcal{PF}, \quad (1)$$

with  $\gcd(x_i, y_i) = 1$  and  $x_i, y_i \geq 2$ , for  $i = 1, \dots, s$ , has finitely many solutions  $x_1, y_1, \dots, x_s, y_s$ . Moreover, there exists a computable positive number  $C$  depending only on the polynomials  $f_1, \dots, f_s$  such that all solutions of equation (1) satisfy  $\max(x_1, y_1, \dots, x_s, y_s) < C$ .

We also have the following inhomogeneous variant of theorem 2.

**Theorem 3.** Let  $f_1, \dots, f_s \in \mathbb{Z}[X]$  be  $s \geq 1$  polynomials of positive degrees. Assume that  $f_i(0) \equiv 1 \pmod{2}$  for  $i = 1, \dots, s$ . Let  $a_1, \dots, a_s$  and  $b_1, \dots, b_s$  be  $2s$  odd integers. Then, the equation

$$f_1(a_1 x_1^{y_1} + b_1 y_1^{x_1}) \cdot \dots \cdot f_s(a_s x_s^{y_s} + b_s y_s^{x_s}) \in \mathcal{PF}, \quad (2)$$

with  $\gcd(x_i, y_i) = 1$  and  $x_i, y_i \geq 2$ , for  $i = 1, \dots, s$ , has finitely many solutions  $x_1, y_1, \dots, x_s, y_s$ . Moreover, there exists a computable positive number  $C$  depending only on the polynomials  $f_1, \dots, f_s$  and the  $2s$  numbers  $a_1, b_1, \dots, a_s, b_s$ , such that all solutions of equation (2) satisfy  $\max(x_1, y_1, \dots, x_s, y_s) < C$ .

We conclude with the following computational results:

**Theorem 4.** All solutions of the equation

$$x^y \pm y^x \in \mathcal{PF} \quad \text{with } \gcd(x, y) = 1 \text{ and } x, y \geq 2,$$

satisfy  $\max(x, y) < \exp 177$ .

**Theorem 5.** All solutions of the equation

$$x^y + y^z + z^x = n! \quad \text{with } \gcd(x, y, z) = 1 \text{ and } x, y, z \geq 2,$$

satisfy  $\max(x, y, z) < \exp 518$ .

## 2. Preliminary Results

The proofs of theorems 1-5 use estimations of linear forms in logarithms of algebraic numbers.

Suppose that  $\zeta_1, \dots, \zeta_l$  are algebraic numbers, not 0 or 1, of heights not exceeding  $A_1, \dots, A_l$ , respectively. We assume  $A_m \geq e^e$  for  $m = 1, \dots, l$ . Put  $\Omega = \log A_1 \dots \log A_l$ . Let  $F = \mathbb{Q}[\zeta_1, \dots, \zeta_l]$ . Let  $n_1, \dots, n_l$  be integers, not all 0, and let  $B \geq \max |n_m|$ . We assume  $B \geq e^2$ . The following result is due to Baker and Wüstholz.

**Theorem BW ([1]).** *If  $\zeta_1^{n_1} \dots \zeta_l^{n_l} \neq 1$ , then*

$$|\zeta_1^{n_1} \dots \zeta_l^{n_l} - 1| > \frac{1}{2} \exp(-(16(l+1)d_F)^{2(l+3)} \Omega \log B). \quad (3)$$

In fact, Baker and Würtholz showed that if  $\log \zeta_1, \dots, \log \zeta_l$  are any fixed values of the logarithms, and  $\Lambda = n_1 \log \zeta_1 + \dots + n_l \log \zeta_l \neq 0$ , then

$$\log |\Lambda| > -(16ld_F)^{2(l+2)} \Omega \log B. \quad (4)$$

Now (4) follows easily from (3) via an argument similar to the one used by Shorey *et al.* in their paper [8].

We also need the following  $p$ -adic analogue of theorem BW which is due to van der Poorten.

**Theorem vdP ([7]).** *Let  $\pi$  be a prime ideal of  $F$  lying above a prime integer  $p$ . Then,*

$$\text{ord}_\pi(\zeta_1^{n_1} \dots \zeta_l^{n_l} - 1) < (16(l+1)d_F)^{12(l+1)} \frac{p^{d_F}}{\log p} \Omega(\log B)^2. \quad (5)$$

The following estimations are useful in what follows.

**Lemma 1.** *Let  $n \geq 2$  be an integer, and let  $p \leq n$  be a prime number. Then*

$$(i) \quad n^{n/2} \leq n! \leq n^n. \quad (6)$$

$$(ii) \quad \frac{n}{4(p-1)} \leq \text{ord}_p n! \leq \frac{n}{p-1}. \quad (7)$$

**Proof.** See [6].

**Lemma 2.** (1) *Let  $s \geq 1$  be a positive integer. Let  $C$  and  $X$  be two positive numbers such that  $C > \exp s$  and  $X > 1$ . Let  $y > 0$  be such that  $y < C \log^s X$ . Then,  $y \log y < (C \log C) \log^{s+1} X$ .*

(2) *Let  $s \geq 1$  be a positive integer, and let  $C > \exp(s(s+1))$ . If  $X$  is a positive number such that  $X < C \log^s X$ , then  $X < C \log^{s+1} C$ .*

**Proof.** (1) Clearly,

$$y \log y < C \log^s X (\log C + s \log \log X).$$

It suffices to show that

$$\log C + s \log \log X < \log C \log X.$$

The above inequality is equivalent to

$$\log C(\log X - 1) > s \log \log X.$$

This last inequality is obviously satisfied since  $\log C > s$  and  $\log X > \log \log X + 1$ , for all  $X > 1$ .

(2) Suppose that  $X \geq C \log^{s+1} C$ . Since  $s \geq 1$  and  $C > \exp(s(s+1))$ , it follows that  $C \log^{s+1} C > C > \exp s$ . The function  $\frac{y}{\log^s y}$  is increasing for  $y > \exp s$ . Hence, since  $X \geq C \log^{s+1} C$ , we conclude that

$$\frac{C \log^{s+1} C}{\log^s(C \log^{s+1} C)} \leq \frac{X}{\log^s X} < C.$$

The above inequality is equivalent to

$$\frac{\log^{s+1} C}{(\log C + (s+1) \log \log C)^s} < 1,$$

or

$$\log C < \left(1 + (s+1) \frac{\log \log C}{\log C}\right)^s.$$

By taking logarithms in this last inequality we obtain

$$\log \log C < s \log \left(1 + (s+1) \frac{\log \log C}{\log C}\right) < s(s+1) \frac{\log \log C}{\log C}.$$

This last inequality is equivalent to  $\log C < s(s+1)$ , which contradicts the fact that  $C > \exp(s(s+1))$ .

### 3. The Proofs

**The Proof of Theorem 1.** By  $C_1, C_2, \dots$ , we shall denote computable positive numbers depending only on the numbers  $\alpha$  and  $\beta$ . Let  $d = d_K$ . Let

$$N_K(\alpha x^y + \beta y^x) = p_1^{\delta_1} \cdot \dots \cdot p_k^{\delta_k}$$

where  $2 < p_1 < p_2 < \dots < p_k$  are prime numbers. For  $\mu = 1, \dots, d$ , let  $\alpha^{(\mu)} x^y + \beta^{(\mu)} y^x$  be a conjugate, in  $K$ , of  $\alpha x^y + \beta y^x$ . Fix  $i = 1, \dots, k$ . Let  $\pi$  be a prime ideal of  $K$  lying above  $p_i$ . We use theorem vdp to bound  $\text{ord}_\pi(\alpha^{(\mu)} x^y + \beta^{(\mu)} y^x)$ . We distinguish two cases:

CASE 1.  $p_i \mid xy$ . Suppose, for example, that  $p_i \mid y$ . Since  $(x, y) = 1$ , it follows that  $p_i \nmid x$ . Hence, by theorem vdp,

$$\text{ord}_\pi(\alpha^{(\mu)} x^y + \beta^{(\mu)} y^x) = \text{ord}_\pi(\alpha^{(\mu)} x^y) + \text{ord}_\pi \left(1 - \left(-\frac{\beta^{(\mu)}}{\alpha^{(\mu)}}\right) y^x x^{-y}\right) <$$

$$< C_1 + C_2 \frac{p_i^d}{\log p_i} \log^4 X. \quad (8)$$

where  $C_1 = d \cdot \log_2 N_K(\alpha)$ , and  $C_2$  can be computed in terms of  $\alpha$  and  $\beta$  using theorem vdP.

CASE 2.  $p_i \nmid xy$ . In this case

$$\begin{aligned} \text{ord}_\pi(\alpha^{(\mu)} x^y + \beta^{(\mu)} y^x) &= \text{ord}_\pi(\alpha^{(\mu)} x^y) + \text{ord}_\pi\left(1 - \left(-\frac{\beta^{(\mu)}}{\alpha^{(\mu)}}\right) \cdot \frac{y^x}{x^y}\right) < \\ &< C_1 + C_2 \frac{p_i^d}{\log p_i} \log^4 X. \end{aligned} \quad (9)$$

Combining Case 1 and Case 2 we conclude that

$$\text{ord}_\pi(\alpha^{(\mu)} x^y + \beta^{(\mu)} y^x) < C_3 \frac{p_i^d}{\log p_i} \log^4 X, \quad (10)$$

where  $C_3 = 2 \cdot \max(C_1, C_2)$ . Hence,

$$\delta_i = \text{ord}_{p_i}(N_K(\alpha x^y + \beta y^x)) < C_4 \frac{p_i^d}{\log p_i} \log^4 X. \quad (11)$$

where  $C_4 = dC_3$ . Denote  $p_k$  by  $P$ . Since  $p_i \leq P$  for  $i = 1, \dots, k$ , it follows, by formula (11), that

$$\log(N_K(\alpha x^y + \beta y^x)) \leq \sum_{i=1}^k \delta_i \cdot \log p_i < kC_4 P^d \log^4 X. \quad (12)$$

Clearly  $k \leq \pi(P)$ , where  $\pi(P)$  is the number of primes less than or equal to  $P$ . Combining inequality (12) with the prime number theorem we conclude that

$$\log(N_K(\alpha x^y + \beta y^x)) < C_5 \frac{P^{d+1}}{\log P} \log^4 X. \quad (13)$$

We now use theorem BW to find a lower bound for  $\log(N_K(\alpha x^y + \beta y^x))$ . Suppose that  $X = y$ . For  $\mu = 1, \dots, d$ , we have

$$\begin{aligned} \log(|\alpha^{(\mu)} x^y + \beta^{(\mu)} y^x|) &= \log(|\alpha^{(\mu)} x^y|) + \log\left(\left|1 - \left(-\frac{\beta^{(\mu)}}{\alpha^{(\mu)}}\right) \frac{y^x}{x^y}\right|\right) > \\ &> C_6 + X \log 2 - C_7 \log^3 X. \end{aligned}$$

where  $C_6 = \min(\log |\alpha^{(\mu)}| \mid \mu = 1, \dots, d)$ , and  $C_7$  can be computed using theorem BW. Hence,

$$\log(N_K(\alpha x^y + \beta y^x)) > dC_6 + dX \log 2 - dC_7 \log^3 X. \quad (14)$$



Let  $C_8 = dC_6$ ,  $C_9 = d \log 2$ , and  $C_{10} = dC_7$ . Let also  $C_{11}$  be the smallest positive number such that

$$\frac{1}{2}C_9y > C_{10} \log^3 y - C_8, \quad \text{for } y > C_{11}.$$

Combining inequalities (13) and (14) it follows that

$$C_5 \frac{P^{d+1}}{\log P} \log^4 X > C_8 + C_9X - C_{10} \log^3 X > \frac{1}{2}C_9X, \quad (15)$$

for  $X \geq C_{11}$ . Inequality (15) clearly shows that

$$P > C_{12} \left( \frac{X}{\log^3 X} \right)^{\frac{1}{d+1}}, \quad \text{for } X \geq C_{11}.$$

**The Proof of Theorem 2.** By  $C_1, C_2, \dots$ , we shall denote computable positive numbers depending only on the polynomials  $f_1, \dots, f_s$ . We may assume that  $f_1, \dots, f_s$  are linear forms with algebraic coefficients. Let  $f_i(X, Y) = \alpha_i X + \beta_i Y$  where  $\alpha_i \beta_i \neq 0$ , and let  $K = \mathbb{Q}[\alpha_1, \beta_1, \dots, \alpha_s, \beta_s]$ . Let  $(x_1, y_1, \dots, x_s, y_s)$  be a solution of (1). Equation (1) implies that

$$\prod_{i=1}^s N_K(\alpha_i x_i^{y_i} + \beta_i y_i^{x_i}) = n_1! \cdot \dots \cdot n_k! \quad (16)$$

We may assume that  $2 \leq n_1 \leq n_2 \leq \dots \leq n_k$ . Let  $X = \max(x_i, y_i \mid i = 1, \dots, s)$ . It follows easily, by inequality (10), that

$$\text{ord}_2 \left( \prod_{i=1}^s N_K(\alpha_i x_i^{y_i} + \beta_i y_i^{x_i}) \right) < C_1 \log^4 X. \quad (17)$$

Hence,

$$\sum_{i=1}^k \text{ord}_2 n_i! < C_1 \log^4 X.$$

By lemma 1, it follows that

$$n_k < 4C_1 \log^4 X. \quad (18)$$

On the other hand, by theorem 1, there exists computable constants  $C_{2i}$  and  $C_{3i}$ , such that

$$P \left( N_K(\alpha_i x_i^{y_i} + \beta_i y_i^{x_i}) \right) > C_{2i} \left( \frac{X_i}{\log^3 X_i} \right)^{1/(d_K+1)} \quad (19)$$

whenever  $x_i, y_i \geq 2$ ,  $\gcd(x_i, y_i) = 1$  and  $X_i = \max(x_i, y_i) > C_{3i}$ . Let  $C_2 = \min(C_{2i} \mid i = 1, \dots, s)$  and let  $C_3 = \max(C_{3i} \mid i = 1, \dots, s)$ . Suppose that  $X > C_3$ . From inequality (19) we conclude that

$$P \left( \prod_{i=1}^s N_K(\alpha_i x_i^{y_i} + \beta_i y_i^{x_i}) \right) > C_2 \left( \frac{X}{\log^3 X} \right)^{1/(d_K+1)}. \quad (20)$$

Since  $P \mid \prod_{i=1}^k n_i!$ , it follows that  $P \leq n_k$ . Combining inequalities (18) and (20) we conclude that

$$C_2 \left( \frac{X}{\log^3 X} \right)^{1/(d_K+1)} < 4C_1 \log^4 X. \quad (21)$$

Inequality (21) clearly shows that  $X < C_4$ .

**The Proof of Theorem 3.** By  $C_1, C_2, \dots$ , we shall denote computable positive numbers depending only on the polynomials  $f_1, \dots, f_s$  and on the numbers  $a_1, b_1, \dots, a_s, b_s$ . Let  $(x_1, y_1, \dots, x_s, y_s)$  be a solution of (2). Let  $X_i = \max(x_i, y_i)$ , and let  $X = \max(X_i \mid i = 1, \dots, s)$ . Finally, let

$$f_i(Z) = c_i \prod_{j=1}^{d_i} (Z - \zeta_{i,j}).$$

Let  $K = \mathbb{Q}[\zeta_{i,j} \mid 1 \leq i \leq s, 1 \leq j \leq d_i]$ , and let  $d = [K : \mathbb{Q}]$ ,  $D = \sum_{i=1}^s d_i$ , and  $c = \prod_{i=1}^s c_i$ .

Let  $\pi$  be a prime ideal of  $K$  lying above 2. Let  $Z_i = a_i x_i^{y_i} + b_i y_i^{x_i}$ . We first bound  $\text{ord}_\pi f_i(Z_i)$ . First, notice that  $\text{ord}_\pi(a_i b_i) = 0$ . Moreover, since  $f_i(0) \equiv 1 \pmod{2}$ , it follows that  $\text{ord}_\pi(\zeta_{i,j}) = 0$ , for all  $j = 1, \dots, d_i$ . We distinguish 2 cases:

CASE 1. Assume that  $2 \nmid x_i y_i$ . Then  $f_i(Z_i) \equiv f_i(0) \equiv 1 \pmod{2}$ . Hence,  $\text{ord}_\pi f_i(Z_i) = 0$ .

CASE 2. Assume that  $2 \mid x_i$ . In this case,  $\text{ord}_\pi(y) = 0$ . Fix  $j = 1, \dots, d_i$ . Then,

$$\text{ord}_\pi(Z_i - \zeta_{i,j}) = \text{ord}_\pi(a_i x_i^{y_i} + (b_i y_i^{x_i} - \zeta_{i,j})). \quad (22)$$

Since  $\text{ord}_\pi(b_i y_i^{x_i}) = \text{ord}_\pi(\zeta_{i,j}) = 0$ , it follows, by theorem vdP, that

$$\text{ord}_\pi(b_i y_i^{x_i} - \zeta_{i,j}) = \text{ord}_\pi(b_i y_i^{x_i} (\zeta_{i,j})^{-1} - 1) < C_1 \log^3 X_i. \quad (23)$$

We distinguish 2 cases:

CASE 2.1.  $y_i \geq C_1 \log^3 X_i$ . In this case, from formula (22) and inequality (23), it follows that

$$\text{ord}_\pi(Z_i - \zeta_{i,j}) = \text{ord}_\pi(b_i y_i^{x_i} - \zeta_{i,j}) < C_1 \log^3 X_i. \quad (24)$$

CASE 2.2.  $y_i < C_1 \log^3 X_i$ . In this case,

$$\text{ord}_\pi(Z_i - \zeta_{i,j}) = \text{ord}_\pi(b_i y_i^{x_i} + (a_i x_i^{y_i} - \zeta_{i,j})). \quad (25)$$

Let  $\Delta = a_i x_i^{y_i} - \zeta_{i,j}$ . Let  $H(\Delta)$  be the height of  $\Delta$ . Clearly,

$$H(\Delta) < C_2 x_i^{d_i y_i}.$$

Hence,

$$\log(H(\Delta)) < \log C_2 + d_i y_i \log x_i < C_3 + C_4 \log^4 X_i,$$

where  $C_3 = \log C_2$ , and  $C_4 = C_1 \cdot \max(d_i \mid i = 1, \dots, s)$ . Since  $\text{ord}_\pi(b_i) = \text{ord}_\pi(y_i^{x_i}) = 0$ , it follows, by theorem vdP, that

$$\begin{aligned} \text{ord}_\pi(Z_i - \zeta_{i,j}) &= \text{ord}_\pi(1 - b_i^{-1} y_i^{-x_i} \Delta) < C_5 \log y_i \log(H(\Delta)) \log^2 x_i < \\ &< C_5 \log^3 X_i (C_3 + C_4 \log^4 X_i). \end{aligned} \quad (26)$$

Let  $C_6 = 2C_4 C_5$ . Also, let

$$C_7 = \exp((C_3/C_4)^{1/4}).$$

From inequalities (23) and (26), it follows that

$$\text{ord}_\pi(Z_i - \zeta_{i,j}) < C_6 \log^7 X, \quad \text{for } X > C_7. \quad (27)$$

Hence,

$$\text{ord}_2\left(\prod_{i=1}^s f_i(Z_i)\right) < C_8 \log^7 X, \quad \text{for } X > C_7, \quad (28)$$

where  $C_8 = 2 \max(sDC_6, c)$ . Suppose now that

$$\prod_{i=1}^s f_i(Z_i) = \prod_{j=1}^k n_j!, \quad (29)$$

where  $2 \leq n_1 \leq n_2 \leq \dots \leq n_k$ . From inequality (28) and lemma 1, it follows that

$$\sum_{j=1}^k n_j < C_9 \log^7 X,$$

where  $C_9 = 4C_8$ . Hence,

$$\begin{aligned} \log\left(\prod_{j=1}^k n_j!\right) &= \sum_{j=1}^k \log n_j! < \sum_{j=1}^k n_j \log n_j < \left(\sum_{j=1}^k n_j\right) \log\left(\sum_{j=1}^k n_j\right) < \\ &< C_9 \log^7 X (\log C_9 + 7 \log \log X), \quad \text{for } X > C_7. \end{aligned} \quad (30)$$

Let  $C_{10}$  be the smallest positive number  $\geq C_7$  such that

$$y > \log C_9 + 7 \log \log y, \quad \text{for } y > C_{10}.$$

From inequality (30), it follows that

$$\log\left(\prod_{j=1}^k n_j!\right) < C_9 \log^8 X, \quad \text{whenever } X > C_{10}. \quad (31)$$

We now bound  $\log\left(\prod_{i=1}^s f_i(Z_i)\right)$ . Fix  $i = 1, \dots, s$ . Suppose that  $y_i = X_i$ . By Theorem BW,

$$\begin{aligned} \log |Z_i| &= \log |a_i x_i^{y_i} + b_i y_i^{x_i}| = \log(|a_i| x_i^{y_i}) + \log \left( \left| 1 - \left(-\frac{b_i}{a_i}\right) y_i^{x_i} x_i^{-y_i} \right| \right) > \\ &> C_{11} + X_i \log 2 - C_{12} \log^3 X_i, \end{aligned} \quad (32)$$

where  $C_{11} = \min(|a_i| \mid i = 1, \dots, s)$ , and  $C_{12}$  can be computed using theorem BW. Let  $C_{13} = (\log 2)/2$ , and let  $C_{14}$  be the smallest positive number  $\geq C_{10}$  such that

$$C_{11} + y \log 2 - C_{12} \log^3 y > C_{13} y, \quad \text{for } y > C_{14}.$$

From inequality (32) it follows that

$$\max(\log |Z_i|) > C_{13} X, \quad \text{for } X > C_{14}. \quad (33)$$

On the other hand, for each  $i = 1, \dots, s$ , there exists two computable constants  $C_i$  and  $C'_i$  such that

$$|f_i(Z_i)| > C_i |Z_i|^{d_i}, \quad \text{whenever } |Z_i| > C'_i.$$

Let  $C_{15} = \min(C_i \mid i = 1, \dots, s)$ , and let  $C_{16} = \max(C'_i \mid i = 1, \dots, s)$ . Finally, let  $C_{17} = \max(C_{14}, (\log C_{16})/C_{13})$ . Suppose that  $X > C_{17}$ . Since  $|f_i(Z_i)| \geq 1$ , for all  $i = 1, \dots, s$ , it follows, by inequality (33), that

$$\log\left(\prod_{i=1}^s f_i(Z_i)\right) \geq \max(\log |f_i(Z_i)| \mid i = 1, \dots, s) >$$

$$> \log C_{15} + \max (\log |Z_i| \mid i = 1, \dots, s) > \log C_{15} + C_{13}X, \quad \text{for } X > C_{17}. \quad (34)$$

From equation (29) and inequalities (31) and (34), it follows that

$$\log C_{15} + C_{13}X < C_9 \log^8 X, \quad \text{for } X > C_{17}. \quad (35)$$

Inequality (35) clearly shows that  $X < C_{18}$ .

**The Proof of Theorem 4.** Let  $X = \max (x, y)$ . Notice that if  $x^y \pm y^x \in \mathcal{PF}$ , then  $xy$  is odd. Hence, by theorem vDP,

$$\text{ord}_2(x^y \pm y^x) = \text{ord}_2(1 - (\mp y)^x x^{-y}) < 48^{36} \cdot \frac{2}{\log 2} \cdot \log^4 X. \quad (36)$$

Suppose that

$$x^y \pm y^x = n_1! \cdot \dots \cdot n_k!, \quad (37)$$

where  $2 \leq n_1 \leq \dots \leq n_k$ . From inequality (36) and lemma 1 it follows that

$$\sum_{i=1}^k n_i \leq 4 \left( \sum_{i=1}^k \text{ord}_2(n_i!) \right) < 48^{36} \cdot \frac{8}{\log 2} \cdot \log^4 X < 12 \cdot 48^{36} \cdot \log^4 X. \quad (38)$$

It follows, by lemma 2 (1), that

$$\begin{aligned} \log(x^y \pm y^x) &= \log \prod_{i=1}^k n_i! = \sum_{i=1}^k \log n_i! < \sum_{i=1}^k n_i \log n_i < \\ &< \left( \sum_{i=1}^k n_i \right) \log \left( \sum_{i=1}^k n_i \right) < 12 \cdot 48^{36} \log(12 \cdot 48^{36}) \cdot \log^5 X < 1703 \cdot 48^{36} \log^5 X. \end{aligned} \quad (39)$$

Suppose now that  $X = y$ . Then, by theorem BW,

$$\begin{aligned} \log |x^y \pm y^x| &\geq \log |x^y - y^x| = \log(x^y) + \log |1 - y^x x^{-y}| > \\ &> X \log 3 - \log 2 - 48^{10} \log^3 X. \end{aligned} \quad (40)$$

Combining inequalities (39) and (40), we conclude that

$$X < X \log 3 < \log 2 + 48^{10} \log^3 X + 1703 \cdot 48^{36} \log^5 X < 1704 \cdot 48^{36} \log^5 X. \quad (41)$$

Let  $C = 1704 \cdot 48^{36}$ , and let  $s = 5$ . Since  $\log C = \log 1704 + 36 \log 48 > 30$ , it follows, by lemma 2 (2), that

$$X < C \cdot \log^6 C < 1704 \cdot 48^{36} \cdot 147^6. \quad (42)$$

Hence,  $\log X < 177$ .

**The Proof of Theorem 5.** Suppose that  $(x, y, z, n)$  is a solution of  $x^y + y^z + z^x = n!$ , with  $\gcd(x, y, z) = 1$  and  $\min(x, y, z) > 1$ . Let  $X = \max(x, y, z)$ . We assume that  $\log X > 519$ . Clearly, not all three numbers  $x, y, z$  can be odd. We may assume that  $2 \mid x$ . In this case, both  $y$  and  $z$  are odd. By theorem vdP,

$$\text{ord}_2(y^z + z^x) = \text{ord}_2(1 - (-y)^{-z} z^x) < 48^{36} \frac{2}{\log 2} \log^4 X < 3 \cdot 48^{36} \log^4 X. \quad (43)$$

We distinguish two cases:

CASE 1.  $y \geq 3 \cdot 48^{36} \log^4 X$ . In this case, by lemma 1,

$$n/4 \leq \text{ord}_2 n! = \text{ord}_2(x^y + y^z + z^x) = \text{ord}_2(y^z + z^x) < 3 \cdot 48^{36} \log^4 X. \quad (44)$$

Hence,

$$n < 12 \cdot 48^{36} \log^4 X. \quad (45)$$

By lemma 2 (1), it follows that

$$n \log n < 12 \cdot 48^{36} \log(12 \cdot 48^{36}) \log^5 X < 1703 \cdot 48^{36} \log^5 X. \quad (46)$$

We conclude that

$$X \log 2 < \log(x^y + y^z + z^x) = \log n! < n \log n < 1703 \cdot 48^{36} \log^5 X.$$

Let  $C = 1703 \cdot 48^{36} / \log 2$ , and let  $s = 5$ . Since  $\log C > 30$ , it follows, by lemma 2 (2), that

$$X < C \log^6 C < 2457 \cdot 48^{36} \cdot 148^6.$$

Hence,  $\log X < 178$ , which is a contradiction.

CASE 2.  $y < 3 \cdot 48^{36} \log^4 X$ . Let  $p$  be a prime number such that  $p \mid y$ . We first show that  $p \nmid x$ . Indeed, assume that  $p \mid x$ . Since  $\gcd(x, y, z) = 1$ , it follows that  $p \nmid z$ . We conclude that  $p \nmid n!$ , therefore  $n < p$ . Hence,

$$n < p \leq y < 3 \cdot 48^{36} \log^4 X.$$

In particular,  $n$  satisfies inequality (45). From Case 1 we know that  $\log X < 178$ , which is a contradiction.

Suppose now that  $p \nmid x$ . Then, by theorem vdP,

$$\begin{aligned} \text{ord}_p(x^y + z^x) &= \text{ord}_p(1 - (-x)^{-y} z^x) < 48^{36} \frac{p}{\log p} \log^4 X < \\ &< 48^{36} y \log^4 X < 3 \cdot 48^{72} \log^8 X. \end{aligned} \quad (47)$$

We distinguish 2 cases:

CASE 2.1.  $z \geq 3 \cdot 48^{72} \log^8 X$ . In this case, by lemma 2 (1) and inequality (47),

$$\begin{aligned} \frac{n}{4(p-1)} &< \text{ord}_p n! = \text{ord}_p (y^z + (x^y + z^x)) = \\ &= \text{ord}_p (x^y + z^x) < 3 \cdot 48^{72} \log^8 X. \end{aligned}$$

Hence,

$$n < 12(p-1) \cdot 48^{72} \log^8 X < 12y \cdot 48^{72} \log^8 X < 36 \cdot 48^{108} \log^{12} X. \quad (48)$$

From lemma 2 (1) we conclude that

$$\begin{aligned} X \log 2 &< \log(x^y + y^z + z^x) = \log n! < n \log n < \\ &< 36 \cdot 48^{108} \log(36 \cdot 48^{108}) \log^{13} X < 317 \cdot 48^{109} \log^{13} X. \end{aligned} \quad (49)$$

Let  $C = 317 \cdot 48^{109} / \log 2$ , and let  $s = 13$ . Since  $\log C > 182$ , it follows, by lemma 2 (2), that

$$X < C \log^{11} C < 458 \cdot 48^{109} \ln^{14}(458 \cdot 48^{109}) < 458 \cdot 48^{109} \cdot 429^{14}.$$

Hence,  $\log X < 513$ , which is a contradiction.

CASE 2.2.  $z < 3 \cdot 48^{72} \log^8 X$ . By theorem vdP, it follows that

$$\begin{aligned} \text{ord}_2(z^x + (x^y + y^z)) &= \text{ord}_2(1 - (-x^y - y^z)z^{-X}) < \\ &< 48^{36} \frac{2}{\log 2} \log(x^y + y^z) \log^3 X < 3 \cdot 48^{36} \log(x^y + y^z) \log^3 X. \end{aligned} \quad (50)$$

We now bound  $\log(x^y + y^z)$ . Let  $y_1 = 3 \cdot 48^{36} \log^4 X$  and  $z_1 = 3 \cdot 48^{72} \log^8 X$ . Since  $y < y_1$  and  $z < z_1$ , it follows that

$$\log(x^y + y^z) < \log(X^{y_1} + y_1^{z_1}) < \log 2 + \max(y_1 \log X, z_1 \log y_1).$$

Since  $z_1 \log y_1 > z_1 > y_1 \log X$ , it follows that

$$\log(x^y + y^z) < \log 2 + z_1 \log y_1.$$

From lemma 2 (1) we conclude that

$$\begin{aligned} \log(x^y + y^z) &< \log 2 + z_1 \log y_1 = \log 2 + \frac{z_1}{y_1} \cdot (y_1 \log y_1) < \\ &< \log 2 + 48^{36} \log^4 X \cdot \left(3 \cdot 48^{36} \log(3 \cdot 48^{36})\right) \log^5 X < 422 \cdot 48^{72} \log^9 X. \end{aligned} \quad (51)$$

From lemma 1 and inequalities (50) and (51) it follows that

$$n/4 < \text{ord}_2 n! = \text{ord}_2(z^x + (x^y + y^z)) < 1266 \cdot 48^{108} \log^{12} X.$$

Hence,

$$n < 5064 \cdot 48^{108} \log^{12} X.$$

By lemma 2 (1), it follows that

$$\begin{aligned} X \log 2 &< \log(x^y + y^z + z^x) = \log n! < n \log n < \\ &< 5064 \cdot 48^{108} \cdot \log(5064 \cdot 48^{108}) \log^{13} X < 22 \cdot 48^{111} \log^{13} X. \end{aligned}$$

Let  $C = 22 \cdot 48^{111} / \log 2$ , and let  $s = 13$ . Since  $\log C > 182$ , it follows, by lemma 2 (2), that

$$X < C \log^{14} C < 22 \cdot 48^{111} \cdot 433^{14}.$$

Hence,  $\log X < 518$ , which is the final contradiction.

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# ANALYTICAL FORMULAE AND ALGORITHMS FOR CONSTRUCTING MAGIC SQUARES FROM AN ARBITRARY SET OF 16 NUMBERS

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In this paper we seek for an answer on Smarandache type question: may one create the theory of Magic squares  $4 \times 4$  in size without using properties of some concrete numerical sequences? As a main result of this theoretical investigation we adduce the solution of the problem on decomposing the general algebraic formula of Magic squares  $4 \times 4$  into two complete sets of structured and four-component analytical formulae.

## 1 Introduction

In the general case *Magic squares* represent by themselves numerical or analytical square tables, whose elements satisfy a set of definite basic and additional relations. The basic relations therewith assign some constant property for the elements located in the rows, columns and two main diagonals of a square table, and additional relations, assign additional characteristics for some other sets of its elements.

Judging by the given general definition of Magic squares, there is no difficulty in understanding that, in terms of mathematics, the problem on Magic squares consists of the three interrelated problems

- a) elucidate the possibility of choosing such a set of elements which would satisfy both the basic and all the additional characteristics of the relations;
- b) determine how many Magic squares can be constructed from the chosen set of elements;
- c) elaborate the practical methods for constructing these Magic squares.

It is a traditional way to solve all mentioned problems with taking into account concrete properties of the numerical sequences from which the Magic

square numbers are generated. For instance, by using this way problems was solved on constructing different Magic squares of natural numbers<sup>1-5</sup>, prime numbers<sup>6, 7</sup>, Smarandache numbers of the 1st kind<sup>8</sup> and so on. Smarandache type question<sup>9</sup> arises: whether a possibility exists to construct the theory of Magic squares without using properties of concrete numerical sequences. The main goal of this paper is finding an answer on this question with respect to problems of constructing Magic squares  $4 \times 4$  in size. In particular, in this investigation we

a) describe a simple way of obtaining a general algebraic formulae of Magic squares  $4 \times 4$ , required no use of algebraic methods, and explain why in the general case this formula does not simplify the solution of problems on constructing Magic square  $4 \times 4$  (Sect. 2);

b) give a description of a set of invariant transformations of Magic squares  $4 \times 4$  (Sect. 3);

c) adduce a general algorithm, suitable for constructing Magic squares from an arbitrarily given set of 16 numbers (Sect. 4);

d) discuss the problems of constructing Magic squares from the structured set of 16 elements (Sect. 5);

e) solve the problem of decomposing the general algebraic formula of Magic squares  $4 \times 4$  into a complete set of the four-component formulae (Sect. 6).

## 2 Constructing the general algebraic formula of a Magic square $4 \times 4$

A table, presented in Fig. 1(2), consists of two orthogonal diagonal Latin squares, contained symbols  $A, B, C, D$  ( $L_1$ ) and  $a, b, c, d$  ( $L_2$ ). Remind<sup>10, 11</sup> that two Latin squares of order  $n$  are called

a) *orthogonal* if being superimposed these Latin squares form a table whose all  $n^2$  elements are various;

b) *diagonal* if  $n$  different elements are located not only in its rows and columns, but also in its two main diagonals.

It is evident that the table 1(2) is transformed in the analytical formula of a Magic square  $4 \times 4$  when its parameter  $b = 0$ . By using Fig. 1(2) we reveal the law governing the numbers of any Magic square  $4 \times 4$  decomposed in two orthogonal diagonal Latin squares. For this aim we rearrange the sets of the symbols in the two-component algebraic formula 1(2) so as it is shown in Fig. 1(6). Further, a

table 1(6) will be called *additional* one. Such name of the table is justified by the following:

a) the table 1(6), containing the same set of elements as the table 1(2), has more simple structure than the formula 1(2);

b) there exists a simple way of passing from this table to a Magic square 4×4: really, if one considers that Fig. 1(1) represents the enumeration of the cells in the table 1(6), then, for passing from this table to a Magic square it will be sufficient to arrange numbers in the new table 4×4 in the order corresponding to one in the *classical* square 1(5) {the Magic square of natural numbers from 1 to 16 }.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

(1)

$A+c$	$B+b$	$C+d$	$D+a$
$D+d$	$C+a$	$B+c$	$A+b$
$B+a$	$A+d$	$D+b$	$C+c$
$C+b$	$D+c$	$A+a$	$B+d$

$(2-L_1+L_2)$

-	-	-	-
$-w$	-	-	$+w$
$+w$	-	-	$-w$
-	-	-	-

$(3-W)$

$A+c$	$B$	$C+d$	$D+a$
$D+d-w$	$C+a$	$B+c$	$A+w$
$B+a+w$	$A+d$	$D$	$C+c-w$
$C$	$D+c$	$A+a$	$B+d$

$(4-L_1+L_2+W)$

3	5	12	14
16	10	7	1
6	4	13	11
9	15	2	8

(5)

$A$	$A+a$	$A+c$	$A+d$
$B$	$B+a$	$B+c$	$B+d$
$C$	$C+a$	$C+c$	$C+d$
$D$	$D+a$	$D+c$	$D+d$

(6)

$A+w$	$A+a$	$A+c$	$A+d$
$B$	$B+a+w$	$B+c$	$B+d$
$C$	$C+a$	$C+c-w$	$C+d$
$D$	$D+a$	$D+c$	$D+d-w$

(7)

Fig. 1. Constructing the general algebraic formula of a Magic square 4×4.

The more simple construction of the additional table in comparison with the formula 1(2) and the possibility of passing from the additional table to a Magic square suggest solving the analogous problems on constructing the

corresponding additional tables instead of solving the problems on constructing Magic squares. Further we shall always perform this replacement of one problem by another.

It is easy to establish by algebraic methods<sup>12, 13</sup> that the general algebraic formula of Magic square of order 4 contains 8 parameters. Thus it has one parameter less than the two-component algebraic formula, presented in Fig. 1(2) with  $b = 0$ . If one takes it into account, then there appears a natural possibility to seek a form for the general algebraic formula of a Magic square  $4 \times 4$  basing, namely, on this two-component algebraic formula. It seems<sup>7</sup> that for introducing one more parameter in the algebraic formula 1(2) one may add cell-wise this formula to the Magic square, shown in Fig. 1(3) {it can be easily counted that the Magic constant of this square equals zero}. Thus, the general algebraic formula of a Magic square  $4 \times 4$  {see Fig. 1(4)} is obtained as a result of the mentioned operation. Therefore it may be written in the simple analytical form

$$L_1 + L_2 + W. \quad (1)$$

By analysing Fig. 1(7), in which the general formula of Magic square  $4 \times 4$  is presented as the additional table, one may conclude, that the availability of eight but not of seven parameters results in a substantial violation of the simple regularity existing for the elements of the additional table 1(6) and by this reason, changing the problem on constructing a Magic square  $4 \times 4$  by that on constructing the corresponding additional table, will not result in a facilitation of its solution in the general case {passing from the additional table 1(7) to the general algebraic formula of the Magic square  $4 \times 4$  1(4) one may realise by means of the classical square 1(5) in the way mentioned above for the additional table 1(6)}.

### 3 A set of invariant transformations of a Magic square $4 \times 4$

By means of rotations by 90 degrees and mappings relative to the sides one can obtain from any Magic square  $4 \times 4$  seven more new ones {see Fig. 2, from which one can judge on changes of a spatial orientation of a Magic square on the basis of the changes in arrangement of the symbols  $A$ ,  $B$ ,  $C$  and  $D$ }. Besides for  $n \geq 4$  there exist such internal transformations (*M-transformations*) of a Magic square  $n \times n$  (permutations of its rows and columns) by which the assigned set of

$[n/2]\{(2[n/2] - 2)!!\}$  Magic squares  $n \times n$  can be obtained<sup>7</sup> from one square with regard for rotations and mappings, where the symbol  $a!!$  means the product of all natural numbers which, firstly, are not exceeding  $a$ , and, secondly, coincide with it in an evenness;  $[a]$  means the integer part of  $a$ . In particular, if the cells of any Magic square  $4 \times 4$  are enumerated so, as it is shown in Fig. 2(9), and under  $M$ -transformations the specific permutations of the cells of the initial square are meant, then, in this case the all 4 possible  $M$ -transformations of a square  $4 \times 4$  can be represented in the form of four tables, depicted in Fig. 2(9 - 12).

It is evident, that when studying Magic squares, constructed from the same set of elements, it is worthwhile, to consider the only squares which can not be obtained from each other by rotations, mappings and  $M$ -transformations. It is usually said about such a family of Magic squares, that it is assigned with regard for rotations, mappings and  $M$ -transformations.

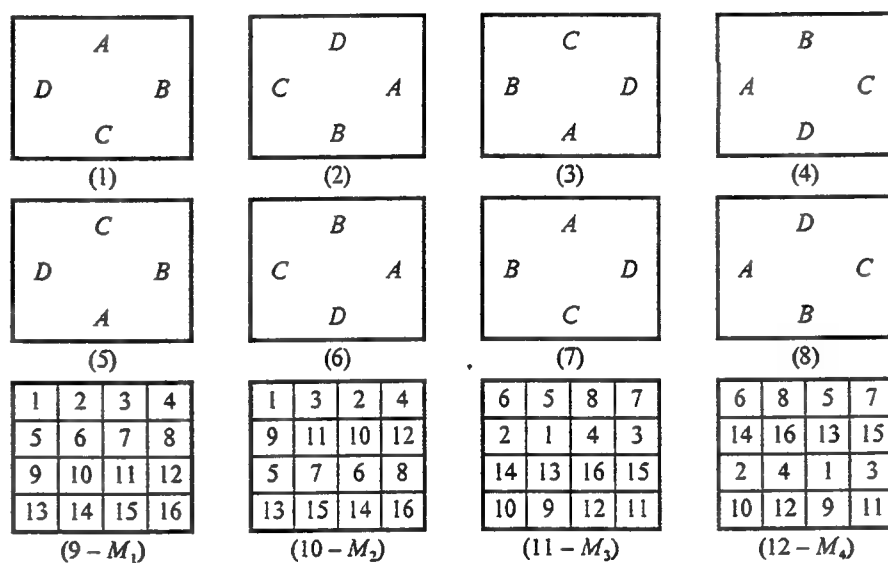


Fig. 2. A set of invariant transformations of a Magic square  $4 \times 4$ .

#### 4 The algorithm for constructing Magic squares from an arbitrary 16 numbers

A complete set of Magic squares  $4 \times 4$  from an arbitrarily given set of 16 numbers with regard for rotations, mappings and  $M$ -transformations one may obtain by the following algorithm<sup>7</sup>:

1. Calculate the sum of all 16 numbers of the given set and, having divided it into 4, obtain the value of the Magic constant  $S$  of the future Magic square  $4 \times 4$ ;
2. Find all possible presentations of the number  $S$  in four different terms each of them belonging to the given set of the numbers;
3. If the number of various partitionings is not smaller than 14, then, using the obtained list of partitionings, form all possible various sets of four Magic rows, containing jointly 16 numbers of the given set;
4. Among the sets of four rows, of the obtained list, find such pairs of the sets which satisfy the following condition: each row of the set has one number from various rows of the other set;
5. It is possible to construct Magic squares  $4 \times 4$  from the above mentioned pairs, if among the earlier found Magic rows (partitionings of the number  $S$ ) one succeeds in finding the two rows such that
  - these rows do not contain identical numbers;
  - each row contains one by one number from various rows both of the first and the second set of the pair.

When constructing Magic squares  $4 \times 4$  from the obtained pairs of the sets consisting of four rows and the sets of the pairs of the rows corresponding to these pairs one should bear in mind that:

- a four-row pair of sets (see point 4) gives a set of Magic rows and columns for a Magic square  $4 \times 4$ ;
- the found pairs of the rows (see point 5) are used for forming the Magic square diagonals;
- if it is necessary to seek for Magic squares with regard for rotations, mappings and  $M$ -transformations, then each differing pair of rows, found for the given pair of sets consisting of four rows, can be utilised for construction of only one Magic square  $4 \times 4$ ;
- the algorithm can be easily realised as a computer program.

## 5 Constructing Magic squares from the structured set of 16 elements

We shall say that a Magic square of order 4 possesses *the structure* (contains a structured set of elements) if it is possible to construct from its elements the eight various pairs of elements with the sum equal to  $1/2$  of the Magic square constant. For obtaining *the structural pattern* of a Magic square, it is sufficient to connect by lines each pair of the elements, forming this structure, directly in the Magic square. The other (*implicit*) way of representing the structural pattern of a Magic square  $4 \times 4$  consists of the following: having chosen 8 various symbols we substitute each pair of numbers, forming the Magic square, by any symbol. As it has been proved by analytical methods<sup>5</sup>, with account for rotations, reflections and *M*-transformations none Magic squares  $4 \times 4$  exist, which contains in its cells 8 even and 8 odd numbers and has structure patterns another than ones shown in the implicit form in Fig. 3(1 – 6). In reality<sup>7, 14</sup> this statement is incorrect because for such Magic squares with respect of invariant transformations there exist 6 more new structure plots, depicted in Fig. 3(7 – 12).

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Fig. 3. A complete set of possible structural patterns in a Magic square  $4 \times 4$ , depicted in the implicit form.

Basing on Fig. 3, for all structural patterns we shall construct a complete set of general structural analytical formulae. Thus, in this section we shall solve the problem *on decomposing* the general algebraic formula 1(4) in the structured ones.

I. Here we present a simple method suitable for constructing general algebraic formulae of Magic squares possessing the structural pattern 3(1 - 4). Besides, we point out some singularities of these four general structured analytical formulae.

As it has been established in Sect. 2 the general algebraic formula of a Magic square  $4 \times 4$  may be represented, as the sum of the two diagonal Latin squares, formed by capital and small Latin letters {see Fig. 1(2)}, and the Magic square {Fig. 1(3)}, having a zero Magic constant. It turns out<sup>7</sup> that general structured algebraic formulae, having structural patterns 3(1 - 4), can be obtained if the required conditions of a structuredness at the fixed structural pattern are written out separately for each of the 3 tables, forming the general algebraic formula 1(4). In particular, diagonal Latin squares 1(2) and the Magic square 1(3) will have structural patterns 3(1 - 4) at the following correlations between their parameters {for convenience, the numbers of the written systems of equations are chosen so that they are identical with the numbers of structural patterns, shown in Fig. 3, by which these equations have been derived}:

$$\begin{array}{llll} 1. A+C=B+D, & 2. A+B=C+D, & 3. A+D=B+C, & 4. A+D=B+C, \\ c = a + d. & a = c + d, & c = a + d, & a = c + d, \\ e = 0. & e = 0. & e = 0. & e = 0. \end{array} \quad (2)$$

Starting from the extracted system of equations (2) one can easily prove that:

1) The cells of an algebraic formula having the structural pattern 3(1) contain two sequences involving elements of the following form:

$$\begin{array}{ll} \text{a) } a_1 + e, & a_1 + a, & a_1 + a + d, & a_1 + d, & a_1 + b, & a_1 + a + b + e, \\ & a_1 + a + b + d, & a_1 + b + d; & & & \\ \text{b) } a_2, & a_2 + a, & a_2 + a + d, & a_2 + d - e, & a_2 + b, & a_2 + a + b, \\ & a_2 + a + b + d - e, & a_2 + b + d. & & & \end{array} \quad (3)$$

One can see from a set of sequences (3) that the regularity existing between the symbols of an general algebraic formula, having structural pattern 3(1), is complicated due to the presence of the four elements containing the symbol  $e$



$a_1 + b + 2c$	$a_1 + b$	$a_2 + c$	$a_2 + 2b + 3c$
$a_2 + b$	$a_2 + b + 2c$	$a_1 + c$	$a_1 + 2b + 3c$
$a_2 + b + 3c$	$a_2 + b + c$	$a_1 + 2b + 2c$	$a_1$
$a_1 + b + c$	$a_1 + b + 3c$	$a_2 + 2b + 2c$	$a_2$

(5)

$a_2$	$a_1 + 2b + c + d$	$a_1 + c$	$a_2 + 2b + 2c + d$
$a_2 + b$	$a_1 + b + 2c + d$	$a_1 + b$	$a_2 + 2c + d$
$a_1 + 2b + 2c + d$	$a_2 + b$	$a_2 + b + 2c + d$	$a_1$
$a_1 + 2c + d$	$a_2 + c$	$a_2 + 2b + c + d$	$a_1 + 2b$

(6)

$a_2 + b + 2c$	$a_1 + b$	$a_2$	$a_1 + 2c$
$a_1 + b + 2c$	$a_2 + c$	$a_1 - b + c$	$a_2 + 2b$
$a_1 - b$	$a_2 + 2b + c$	$a_1 + b + c$	$a_2 + 2c$
$a_2 + b$	$a_1 - b + 2c$	$a_2 + 2b + 2c$	$a_1$

(7)

$a_1 + 2b$	$a_2 + 10b$	$a_1 + 4b$	$a_2 + 4b$
$a_2 + b$	$a_1 + 10b$	$a_2 + 8b$	$a_1 + b$
$a_2 + 9b$	$a_1$	$a_2 + 2b$	$a_1 + 9b$
$a_1 + 8b$	$a_2$	$a_1 + 6b$	$a_2 + 6b$

(8)

$a_1 + 4b$	$a_1 + 12b$	$a_1 + 10b$	$a_1 + 16b$
$a_1 + 11b$	$a_1 + 8b$	$a_1 + 6b$	$a_1 + 17b$
$a_1 + 14b$	$a_1 + 7b$	$a_1 + 21b$	$a_1$
$a_1 + 13b$	$a_1 + 15b$	$a_1 + 5b$	$a_1 + 9b$

(10)

$a_1$	$a_2 + 8b$	$a_2$	$a_1 + 8b$
$a_2 + 6b$	$a_1 + 6b$	$a_1 + 2b$	$a_2 + 2b$
$a_1 + 5b$	$a_2 + b$	$a_2 + 7b$	$a_1 + 3b$
$a_2 + 5b$	$a_1 + b$	$a_1 + 7b$	$a_2 + 3b$

(9)

$a_1 + 12b$	$a_1 + 16b$	$a_1 + 4b$	$a_1 + 10b$
$a_1 + 14b$	$a_1 + 7b$	$a_1 + 21b$	$a_1$
$a_1 + 11b$	$a_1 + 6b$	$a_1 + 8b$	$a_1 + 17b$
$a_1 + 5b$	$a_1 + 13b$	$a_1 + 9b$	$a_1 + 15b$

(11)

$a_1 + 3b$	$(a_1 + a_2)/2 + 3b$	$(a_1 + a_2)/2 - b$	$a_2 + 5b$
$a_2 + 3b$	$a_2 + b$	$a_1 + 5b$	$a_1 + b$
$a_1 + 4b$	$a_1$	$a_2 + 4b$	$a_2 + 2b$
$a_2$	$(a_1 + a_2)/2 + 6b$	$(a_1 + a_2)/2 + 2b$	$a_1 + 2b$

(12)

Fig. 4. General algebraic formulae of a Magic square 4x4 with structural patterns 3(5 - 12).

{as well as in the general algebraic formula of a Magic square  $4 \times 4$  shown in Fig. 1(4)}. Consequently, the knowledge of the regularity existing between the elements of the general algebraic formula with structural pattern 3(1) can not be of help in creating a convenient and practical algorithm for constructing corresponding Magic squares {as well as for the general algebraic formula 1(4)}.

2) The general algebraic formulae having structural patterns 3(2 - 4) are decomposable in sums of two diagonal Latin squares {parameters  $b$  and  $e$  are equal to zero}. Hence, there is the simple regularity for the elements of additional tables of general algebraic formulae with structural patterns 3(2 - 4) and, consequently, the problem on constructing such Magic squares  $4 \times 4$  from a given structured set of 16 elements is easy to solve by means of these three formulae.

II. Taking into account that for structural patterns 3(1 - 4) there exists a simple method for constructing the general algebraic formulae (see point I) we present in Fig. 4 a set of 8 general algebraic formulae which possess only structural patterns of 3(5 - 12) {the form of representing these formulae is chosen so that it reveals the regularity existing between their elements}. Analysing the analytical formulae presented in Fig. 4 we may come to the following conclusions:

1) among the all above formulae, the formulae 10 and 11 have the most simple structure: the set, consisting of their 16 elements, is completely defined by the first element of the sequence  $a_1$  and the value of the parameter  $b$ ;

2) the sets of the symbols, contained in the formulae 5, 6, 7, 8 and 9, may be represented in the form of the two identically constructed sequences consisting of 8 elements {the reader can himself get assured that the same holds true also for general algebraic formulae possessing structural patterns 3(2 - 4)};

3) there are two arithmetical sequences, each containing 6 terms and having the same progression difference in the formula 12. Thus, the complication of the regularity, governing the symbols forming the algebraic formula 12, is caused only by four of its elements {compare with the above information concerning the general algebraic formula possessing structural pattern 3(1)}.

*The main conclusion* which may be drawn from the above written implies that for constructing Magic squares having the structural patterns 3(2 - 12) it is preferable to use the general algebraic formulae of Magic squares  $4 \times 4$ , corresponding to these structural patterns.

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Fig. 5. A set of A-, B-, C-forms, suitable for constructing Magic squares 4x4.

## 6 Four-component algebraic formulae of Magic squares 4×4

1. *Four-component algebraic formulae of the classical Magic squares 4×4.* Since a classical (Magic) square contains in its cells 16 different natural numbers  $N$  ( $1 \leq N \leq 16$ ) then one may write<sup>4, 12</sup> the formula for decomposing the number  $N$  in 5 terms:

$$N = 8a + 4b + 2c + d + 1, \quad (4)$$

where the parameters  $a, b, c$  and  $d$  can assume only two values: either 0 or 1. By means of (4) any classical square 4×4 may be identically decomposed in 4 tables ( $a$ -,  $b$ -,  $c$ -,  $d$ -components) each of them containing 8 zeros and 8 units. From theoretical point of view<sup>4</sup> there exist the only three groups of Magic squares:

1) *correct squares* — all the decomposition tables are by themselves Magic squares: they have in all the rows, columns and in the two main diagonals by 2 zeros and 2 units. Further such decomposition tables we shall denote as *A-form*.

2) *regular squares* — at least one of the decomposition tables differs from correct one by existing at least one of the components of the formula, which is necessarily a regular one: each of its rows and columns contains by two zeros and two units, but this condition being not preserved for the main diagonal. Further such decomposition tables we shall denote as *B-form* if its both main diagonals contain 4 or 0 zeros (units) and *C-form* if its the main diagonals contain 1 or 3 zeros (units).

3) *irregular squares* — at least one of the decomposition tables differs from correct and regular one by existing at least one of the components of the formula has one row or one column where the number of the same symbols of one kind is distinct from two.

As it can be proved by analytical methods

a) by using *A-forms* one may construct<sup>4, 12</sup> the only 11 different algebraic formulae of correct Magic squares and with account for rotations and reflections<sup>7</sup> the only 7 following

$$\begin{aligned} A_1 A_2 A_5 A_6, & \quad A_1 A_2 A_3 A_5, \quad A_1 A_3 A_5 A_7, \quad A_1 A_2 A_6 A_7, \\ A_2 A_3 A_6 A_7, & \quad A_3 A_5 A_6 A_7, \quad A_4 A_5 A_6 A_8, \end{aligned} \quad (5)$$

will be different among them, where  $A_1 - A_8$  forms are presented in Fig. 5(1 – 8);

b) by using *B*- and *C*-forms one may construct<sup>4</sup> with account for rotations, reflections and *M*-transformations the only 15 different algebraic formulae of regular Magic squares

$$\begin{aligned}
 BC\bar{A}A &— B_1C_1A_2A_3, \quad B_1C_2A_1A_4; \\
 BCBA &— B_1C_1B_2A_2, \quad B_1C_1B_3A_2, \quad B_1C_1B_3A_3, \quad B_1C_1B_4A_3, \\
 &\quad B_1C_2B_2A_1, \quad B_1C_2B_3A_4, \quad B_1C_2B_6A_4; \\
 BCBB &— B_1C_1B_2B_3, \quad B_1C_1B_2B_4, \quad B_1C_1B_3B_4, \quad B_1C_2B_2B_5, \\
 &\quad B_1C_2B_2B_6, \quad B_1C_2B_3B_6,
 \end{aligned} \tag{6}$$

where  $C_1 - C_4$  and  $B_1 - B_6$  forms are presented in Fig. 5(9 – 18).

c) for classical squares  $4 \times 4$  the complete set of four-component algebraic formulae consists of algebraic formulae of the only correct and regular Magic squares<sup>7</sup> {see sets of formulae (5) and (6)}.

2. *Four-component algebraic formulae of generalised Magic squares.* Denote, first, *A*-components of a correct Magic square  $4 \times 4$  by the symbols  $F_1, F_2, F_3$  and  $F_4$ ; second, the trivial Magic square, whose 16 cells are filled with units, by the symbol  $E$ . As it follows from point 1, any correct classical square  $4 \times 4$  can be represented as the sum of 5 tables (the first 3 tables should be multiplied by 8, 4 and 2):

$$8F_1 + 4F_2 + 2F_3 + F_4 + E. \tag{7}$$

An algebraic generalisation of this notation is the expression

$$\alpha F_1 + \beta F_2 + \sigma F_3 + \delta F_4 + \varepsilon E, \tag{8}$$

which represents the general recording form of a Magic square  $4 \times 4$  decomposable in the sum of the 4-th *A*-components. Since the numbers of a classical square  $4 \times 4$  may be calculated from the formula (4), the formula (8) obviously permits to find the symbols contained in the cells of the *generalised correct* Magic square  $4 \times 4$ . In particular, there exist the following relations

$$\begin{aligned}
 1 &— \varepsilon, & 5 &— \varepsilon + \beta, & 9 &— \varepsilon + \alpha, & 13 &— \varepsilon + \alpha + \beta, \\
 2 &— \varepsilon + \delta, & 6 &— \varepsilon + \beta + \delta, & 10 &— \varepsilon + \alpha + \delta, & 14 &— \varepsilon + \alpha + \beta + \delta,
 \end{aligned} \tag{9}$$

$$\begin{array}{llll}
3 - \varepsilon + \sigma, & 7 - \varepsilon + \beta + \sigma, & 11 - \varepsilon + \alpha + \sigma, & 15 - \varepsilon + \alpha + \beta + \sigma, \\
4 - \varepsilon + \sigma + \delta, & 8 - \varepsilon + \beta + \sigma + \delta, & 12 - \varepsilon + \alpha + \sigma + \delta, & 16 - \varepsilon + \alpha + \beta + \sigma + \delta.
\end{array}$$

between natural numbers from 1 to 16 and the symbols of the generalised correct Magic square  $4 \times 4$ .

Note that the cells of the table, shown in Fig. 6(1), contain a complete set of the symbols of the generalised correct Magic square  $4 \times 4$ . These symbols are arranged so that the first cell of the table contains the symbol  $\varepsilon$ , the second one contains the symbols  $\varepsilon + \delta$  and so on. Thus, the mentioned table is *additional* by the definition and permits to construct various algebraic formulae of the generalised correct Magic squares of the fourth order for the assigned correct classical squares  $4 \times 4$ .

Change the form of recording the table 6(1) by introducing the new symbols  $g, h$  and  $f$  with the correlations  $g = \varepsilon + \beta, h = \varepsilon + \alpha, f = \varepsilon + \alpha + \beta$ . The new form of the table is shown in Fig. 6(3). The table 6(3) makes it clear that the rows of the initial additional table of the generalised correct Magic square  $4 \times 4$  contain the sequences of four numbers formed by the same regularity. Let it be also noted, that the new table (as it may be easily verified) completely corresponds to the initial one only if between its parameters  $\varepsilon, g, h$  and  $f$  the correlation  $\varepsilon + f = g + h$  is fulfilled. Thus, for constructing concrete examples of the generalised correct Magic squares it is necessary to continue the search for the indicated sequences involving four numbers until one finds among their first terms the two pairs of numbers having the same sum.

For example, the generalised correct Magic square  $4 \times 4$  may be formed from the following eight pairs of prime numbers "the twins": 29, 31; 59, 61; 71, 73; 101, 103; 197, 199; 227, 229; 239, 241; 269, 271.

The additional table, shown in Fig. 6(1) one also may use for constructing the algebraic formulae of the *generalised regular* Magic squares on the basis of the given classical squares  $4 \times 4$ . However, due to the fact that the condition of Magicity is not fulfilled on the diagonals of regular tables (see point 1) for obtaining algebraic formulae of Magic squares in this case one has to assign additional correlations between the parameters of the additional table. For the set (6) of regular formulae of the Magic square  $4 \times 4$  these necessary correlations between the parameters of the additional table 6(1) have the form, depicted in Fig. 6(6).

$e$	$e+d$	$e+c$	$e+c+d$
$e+b$	$e+b+d$	$e+b+c$	$e+b+c+d$
$e+a$	$e+a+d$	$e+a+c$	$e+a+c+d$
$e+a+b$	$e+a+b+d$	$e+a+b+c$	$e+a+b+c+d$

(1)

17	29	41	53
47	59	71	83
227	239	251	263
257	269	281	293

(2)

$e$	$e+d$	$e+c$	$e+c+d$
$g$	$g+d$	$g+c$	$g+c+d$
$h$	$h+d$	$h+c$	$h+c+d$
$f$	$f+d$	$f+c$	$f+c+d$

(3)

1	7	67	73
37	43	103	109
157	163	223	229
193	197	257	263

(4)

83	113	293	503
41	71	251	461
281	311	491	701
239	269	449	659

(5)

Components of formula				Correlations between the parameters
$a$	$b$	$c$	$d$	
$A$	$A$	$B$	$C$	$c = 2d,$
$B$	$C$	$A$	$A$	$a = 2b,$
$A$	$B$	$C$	$A$	$b = 2c,$
$A$	$B$	$B$	$C$	$b = c + 2d,$
$B$	$B$	$C$	$A$	$a = b + 2c,$
$B$	$B$	$B$	$C$	$a = c + b + 2d.$

(6)

Fig. 6. Examples of constructing additional tables for the generalised correct (1-5) and regular (2 -  $AABC$ , 4 -  $ABBC$ , 5 -  $BBBC$ ) Magic squares  $4 \times 4$ .

It is noteworthy, that for the given type of a four-component regular formula the set of the symbols, positioned in the cells of additional tables of the generalised regular Magic squares  $4 \times 4$ , does not depend upon the form of additional correlations between the parameters of the additional table 6(1), in other words, it does not depend on the position of the  $C$ -form in the  $a$ -,  $b$ -,  $c$ -,  $d$ -decompositions of regular formulae. One can be immediately convinced in this by constructing on the basis of Fig. 6(1) all six additional tables for various algebraic formulae of the generalised regular Magic squares  $4 \times 4$ . Thus, if it is possible to construct one additional table for algebraic formulae of the type

*AABC* or *ABBC* from the given set involving arbitrary 16 numbers, then it is also possible to construct the other additional tables of Magic squares of the given type, distinct from the above constructed one by the form of additional conditions for the parameters of the table 6(1). With regard for the above stated, only 3 additional tables, filled with prime numbers, for which the reader is referred to Fig. 6(2, 4, 5), suffice for constructing a complete family of different regular Magic squares  $4 \times 4$ .

1367	1468	2358	2457
1457	1458	2368	2367
1368	1467	2357	2458
1358	2467	1357	2468

(1)

2368	1467	2357	1458
2367	1457	1358	2468
2467	1357	1368	2458
1367	1468	2358	2457

(2)

1367	1458	2368	2457
1457	1468	2358	2367
1358	1467	2357	2468
1368	2467	1357	2458

(3)

1367	2368	1458	2457
2367	2358	1468	1457
2468	2357	1467	1358
1368	1357	2467	2458

(4)

1367	1468	2358	2457
1467	1458	2368	2357
1358	1457	2367	2468
1368	2467	1357	2458

(5)

1368	2367	2458	1457
1367	2358	2467	1458
2468	1358	1467	2357
2368	1357	1468	2457

(6)

Fig. 7. Examples of irregular four-component algebraic formulae of Magic squares  $4 \times 4$ .

In conclusion of this section we would like to draw attention that with regard for rotations, mappings and *M*-transformations there exist<sup>7</sup> 81 irregular four-component algebraic formulae of Magic squares  $4 \times 4$ . For instance, 6 formulae of such type are presented in Fig. 7 {for splitting the formulae, shown in Fig. 7, in four components, it suffices to retain in the formulae, at first, only the digits 1 and 2 (1st component), and then, only the digits 3 and 4 (2nd component), etc.}. Hence, the solution of the problem *on decomposing* the general algebraic formula 1(4) into a complete set of the four-component ones



has following form: there are 7 formulae for correct Magic squares  $4 \times 4$  {with account for rotations and reflections}, 15 and 81 formulae correspondingly for regular and irregular Magic squares  $4 \times 4$  {with account for rotations, mappings and  $M$ -transformations}. Thus, it is *the main conclusion* of this section that the complete set of four-component analytical formulae of Magic squares  $4 \times 4$  can not simplify the solution of the problem on constructing Magic squares  $4 \times 4$  from an arbitrary given set of 16 numbers but it can make so for constructing the generalised correct and regular Magic squares  $4 \times 4$ .

## 7 Summary

As it have been demonstrated in this paper discussed Smarandache type question – whether a possibility exists to construct the theory of Magic squares without using properties of concrete numerical sequences – has the positive answer. However, to construct this theory for Magic squares  $4 \times 4$  in size, the new type of mathematical problems was necessary to introduce. Indeed, in terms of algebra, any problems on constructing Magic squares without using properties of concrete numerical sequences may be formulated as ones on composing and solving the corresponding systems of algebraic equations. Thus, algebraic methods can be applied for

- a) constructing the algebraic formulae of Magic squares;
- b) finding the transformations translating an algebraic formula of a Magic square from one form into another one;
- c) elucidating the general regularities existing between the elements of Magic squares;
- d) finding for an algebraic formula of a Magic square, containing  $m$  freely chosen parameters, the equivalent set consisting of  $L$  algebraic formulae each containing the number of freely chosen parameters less than  $m$ .

The new for algebra the type of mathematical problems is presented in points (b) – (d). It is evident that without introducing these problems the algebraic methods are not effective themselves. For instance, in the common case (see Sect. 2) the general formula of Magic square  $4 \times 4$  can not simplify the solution of problems on constructing Magic squares  $4 \times 4$  from an arbitrary given set of 16 numbers. In particular, even when solution of discussed problems is sought by means of a computer, in calculating respect it is more preferable for obtaining the solution to use algorithm, described in Sect. 4, than one, elaborated on the base of the general formula of Magic square  $4 \times 4$ . But by

means of decomposing the general algebraic formula of Magic squares  $4 \times 4$  into complete sets of a defined type of analytical formulae one may decrease the common amount of freely chosen parameters in every such formula and, consequently, substantially simplify the regularity existing for the elements of every formula. In other words, for constructing Magic squares  $4 \times 4$  from an arbitrary given set of 16 numbers there appears a peculiar possibility of using the set algebraic formulae with more simple structure instead of use one complex algebraic formula Magic square  $4 \times 4$ .

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## Palindromic Numbers And Iterations of the Pseudo-Smarandache Function

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In his delightful little book[1] Kenichiro Kashara introduced the Pseudo-Smarandache function.

**Definition:** For any  $n \geq 1$ , the value of the Pseudo-Smarandache function  $Z(n)$  is the smallest integer  $m$  such that  $n$  evenly divides

$$\sum_{k=1}^m k.$$

And it is well-known that the sum is equivalent to  $\frac{m(m+1)}{2}$ .

Having been defined only recently, many of the properties of this function remain to be discovered. In this short paper, we will tentatively explore the connections between  $Z(n)$  and a subset of the integers known as the palindromic numbers.

**Definition:** A number is said to be a palindrome if it reads the same forwards and backwards. Examples of palindromes are

121, 34566543, 111111111

There are some palindromic numbers  $n$  such that  $Z(n)$  is also palindromic. For example,

$$Z(909) = 404 \quad Z(2222) = 1111$$

In this paper, we will not consider the trivial cases of the single digit numbers.

A simple computer program was used to search for values of  $n$  satisfying the above criteria. The range of the search was,  $10 \leq n \leq 10000$ . Of the 189 palindromic values of  $n$  within that range, 37, or slightly over 19%, satisfied the criteria.

Furthermore it is sometimes possible to repeat the function again and get another palindrome.

$$Z(909) = 404, \quad Z(404) = 303$$

and once again, a computer program was run looking for values of  $n$  within the range

$1 \leq n \leq 10,000$ . Of the 37 values found in the previous test, 9 or slightly over 24%, exhibited the above properties.

Repeating the program again, looking for values of  $n$  such that  $n$ ,  $Z(n)$ ,  $Z(Z(n))$  and  $Z(Z(Z(n)))$  are all palindromic, we find that of the 9 found in the previous test, 2 or roughly 22%, satisfy the new criteria.

**Definition:** Let  $Z^k(n) = Z(Z(Z(\dots(n))))$  where the  $Z$  function is executed  $k$  times. For notational purposes, let  $Z^0(n) = n$ .

Modifying the computer program to search for solutions for a value of  $n$  so that  $n$  and all iterations  $Z^i(n)$  are palindromic for  $i = 1, 2, 3$  and  $4$ , we find that there are no solutions in the range  $1 \leq n \leq 10,000$ . Given the percentages already encountered, this should not be a surprise. In fact, by expanding the search up through  $100,000$  one solution was found.

$$Z(86868) = 17271, Z(17271) = 2222, Z(2222) = 1111, Z(1111) = 505$$

Since  $Z(505) = 100$ , this is the largest such sequence in this region.

Computer searches for larger such sequences can be more efficiently carried out by using only palindromic numbers for  $n$ .

**Unsolved Question:** What is the largest value of  $m$  so that for some  $n$ ,  $Z^k(n)$  is a palindrome for all  $k = 0, 1, 2, \dots, m$ ?

**Unsolved Question:** Do the percentages discussed previously accurately represent the general case?

Of course, an affirmative answer to the second question would mean that there is no largest value of  $m$  in the first.

**Conjecture:** There is no largest value of  $m$  such that for some  $n$ ,  $Z^k(n)$  is a palindrome for all  $k = 0, 1, 2, 3, \dots, m$ .

There are solid arguments in support of the truth of this conjecture. Palindromes tend to be divisible by palindromic numbers, so if we take  $n$  palindromic, many of the numbers that it divides would also be palindromic. And that palindrome is often the product of two numbers, one of which is a different palindrome. Numbers like the repunits,  $11 \dots 111$  and those with only a small number of different digits, like  $1001$  and  $505$  appeared quite regularly in the computer search.

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# *Computational Aspect of Smarandache's Function*

Sabin Tabirca

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**Abstract:** *The note presents an algorithm for the Smarandache's function computation. The complexity of algorithm is studied using the main properties of function. An interesting inequality is found giving the complexity of the function on the set  $\{1, 2, \dots, n\}$ .*

## **1. Introduction**

In this section, the main properties of function are reviewed. The Smarandache's function notion reported for the first time in [1]. The main results concerning the function can be found in [1], [2].

The function  $S: N \rightarrow N$  defined by  $S(n) = \min\{k | k! \leq n\}$  is called Smarandache's function. This concept is connected with the prime number concept, because using the prime numbers an expression for the function is given. The important properties that are used in this paper, are showed bellow.

1. For all  $n \in N$ , the inequality  $S(n) \leq n$  is true. When  $n$  is a prime number, the equality is obtained .
2. If  $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_m^{k_m}$  is the prime number decomposition then
$$S(n) = \max\{S(p_1^{k_1}), S(p_2^{k_2}), \dots, S(p_m^{k_m})\} . \quad (1)$$
3. The inequality  $S(p^k) \leq p \cdot k$  is true, if  $p$  is a prime number. (2)

A lot of conjectures or open problems related to the Smarandache's function appear in the number theory. Several such problems have been studied using computers and reported in relevant literature, e.g. [3], [4]. Using the computer

## 2. An Algorithm for the Smarandache function

In the following, an algorithms for computing the function  $S$  is presented. The main idea of the algorithm is to avoid the calculations of factorial, because the values of  $n!$ , for  $n > 10$  are a very big number and cannot be calculated using a computer.

Let  $(x_k)_{k \geq 1}$  a sequence of integer numbers defined by  $x_k = k! \bmod n$ ,  $(\forall) n > 0$ . Using the definition of sequence, the following equations can be written:

$$\bullet \quad x_1 = 1 \quad (3)$$

$$\bullet \quad x_{k+1} = (k+1)! \bmod n = (k+1)k! \bmod n = (k+1)x_k \bmod n. \quad (4)$$

Based on the linear equation (4),  $S$  can be calculated as  $S(n) = \min\{k | k! \bmod n = 0\}$ .

The algorithm for  $S(n)$  performs the computation of  $x_1, x_2, \dots, x_k$  until the 0 value is found. The PASCAL description of this algorithm is given bellow.

```
function S(n:integer):integer;
var
  k,x:integer;
begin
  x:=1;k:=1;
  while x<>0 do
    begin
      x:=x*(k+1) mod n;
      k:=k+1;
    end;
  s:=k;
end;
```

An analysis for the complexity of algorithm is presented in the following. The work-case complexity and the average complexity are studied.

**Theorem 2.1**

If  $n = p_1^{k_1} \cdot p_2^{k_2} \dots p_m^{k_m}$  then the complexity of the algorithm for computing  $S(n)$  is  $O(\max\{p_1 k_1, p_2 k_2, \dots, p_m k_m\})$ .

**Proof**

The algorithm computes the value  $S(n)$  generating the sequence  $x_1, x_2, \dots, x_{S(n)}$ . The number of operation has the complexity  $O(S(n))$ .

Based on (1) and (2), the following inequality holds

$$S(n) = \max\{S(p_1^{k_1}), S(p_2^{k_2}), \dots, S(p_m^{k_m})\} \leq \max\{p_1 k_1, p_2 k_2, \dots, p_m k_m\}. \quad (5)$$

Therefore, it can be concluded that the complexity of computing  $S(n)$  is  $O(\max\{p_1 k_1, p_2 k_2, \dots, p_m k_m\})$ .

Other aspect of complexity is given by the average operations number. Assume the value  $S(k)$  is generated, where  $k$  is a number between 1 and  $n$ . This value can be computed with  $S(k)$  operations. Therefore the process takes  $S(1)$  operations for the value  $S(1)$ ,  $S(2)$  operations for the value  $S(2)$ , ..., a.s.o. The average of the numbers operations is  $\bar{S} = \frac{1}{n} \sum_{i=1}^n S(i)$  and gives an other estimation for the complexity of algorithm.

In the following, a possible upper bounds for  $\bar{S}$  are sought. Obviously,  $\bar{S}$  verifies the simple inequality

$$\bar{S} = \frac{1}{n} \sum_{i=1}^n S(i) \leq \frac{1}{n} \sum_{i=1}^n i = \frac{n+1}{2} = \frac{1}{2}n + \frac{1}{2}. \quad (6)$$

Inequality (6) can be improved using the strong inequalities for  $S(i)$ .

**Lemma 2.1.**

If  $i > 2$  is an integer number the following inequalities hold

$$1. S(2i) + S(2i+1) \leq 3i+1. \quad (7)$$

$$2. S(2i-1) + S(2i) \leq 3i-1. \quad (8)$$

**Proof**

Assuming  $i > 2$  follow  $S(2i) \leq i$ . Applying this result both inequality are true.

Based on lemma 2.1, we can found an upper bound for the  $\bar{S}$  better than in (6).

### Theorem 2.2

If  $n > 5$  is a integer number then the inequality  $\bar{S} = \frac{1}{n} \sum_{i=1}^n S(i) \leq \frac{3}{8}n + \frac{1}{4} + \frac{2}{n}$  is true.

#### Proof

Two cases are considered to prove the inequality.

*Case 1:  $n=2m$*

$$n\bar{S} = \sum_{i=1}^n S(i) = \sum_{i=1}^{n/2} [S(2i-1) + S(2i)] = S(1) + S(2) + S(3) + S(4) + \sum_{i=3}^{n/2} [S(2i-1) + S(2i)]$$

Based on (8) it can be derived that

$$\begin{aligned} n\bar{S} &= 9 + \sum_{i=3}^{n/2} [S(2i-1) + S(2i)] \leq 9 + \sum_{i=3}^{n/2} (3i-1) = 2 + \sum_{i=1}^{n/2} (3i-1) = 2 + \frac{3}{2} \frac{n}{2} \left( \frac{n}{2} + 1 \right) - \frac{n}{2} = \\ &= 2 + \frac{3}{2} \frac{n}{2} \left( \frac{n}{2} + 1 \right) - \frac{n}{2} = \frac{3}{8}n^2 + \frac{1}{4}n + 2. \end{aligned}$$

$$\text{Dividing by } n, \text{ we obtain the inequality } \bar{S} \leq \frac{3}{8}n + \frac{1}{4} + \frac{2}{n}. \quad (9)$$

*Case 2:  $n=2m+1$*

$$n\bar{S} = \sum_{i=1}^n S(i) = S(1) + \sum_{i=1}^{(n-1)/2} [S(2i) + S(2i+1)] = S(2) + S(3) + S(4) + S(5) + \sum_{i=3}^{(n-1)/2} [S(2i) + S(2i+1)]$$

Using (7), it is found

$$\begin{aligned} n\bar{S} &= 14 + \sum_{i=3}^{(n-1)/2} [S(2i) + S(2i+1)] \leq 14 + \sum_{i=3}^{(n-1)/2} (3i+1) = 3 + \sum_{i=1}^{(n-1)/2} (3i+1) = 3 + \frac{3}{2} \frac{n-1}{2} \frac{n+1}{2} + \frac{n-1}{2} = \\ &= 3 + \frac{3}{8}(n^2 - 1) + \frac{n-1}{2} = \frac{3}{8}n^2 + \frac{1}{2}n + \frac{17}{8}. \text{ Thus, } \bar{S} \leq \frac{3}{8}n + \frac{1}{2} + \frac{17}{8n}. \end{aligned} \quad (10)$$

From (9) and (10), it is found  $\bar{S} \leq \frac{3}{8}n + \min\{\frac{1}{4} + \frac{2}{n}, \frac{1}{2} + \frac{17}{8n}\} \leq \frac{3}{8}n + \frac{1}{4} + \frac{2}{n}$ .

### 3. Final Remarks



1. Based on theorem 2.2 we can say that the average operations number for computing the Smarandache's function is less than  $\frac{3}{8}n + \frac{1}{4} + \frac{2}{n}$ .
2. The upper bound  $\frac{3}{8}n + \frac{1}{4} + \frac{2}{n}$  improves the previous bound  $\frac{1}{2}n + \frac{1}{2}$ .
3. The improving process can be extended using other sort of inequalities give the prime numbers 2 and 3. A lemma as similar as lemma 2.1 finds the upper bounds the sum of sixth consecutive terms of Smarandache's function.
4. Using the algorithm for computing the function S, the Smarandache's function can be tabulated. The values  $S(1), \dots, S(n)$  for all  $n < 5000$  can be found. The algorithm should be reviewed to be able to compute the Smarandache function for the big numbers.

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# THE ANALYTICAL FORMULAE YIELDING SOME SMARANDACHE NUMBERS AND APPLICATIONS IN MAGIC SQUARES THEORY

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In this paper we study the properties of some six numerical Smarandache sequences. As result we present a set of analytical formulae for the computation of numbers in these Smarandache series and for constructing Magic squares  $3 \times 3$  in size from  $k$ -truncated Smarandache numbers. The examples of Magic squares  $3 \times 3$  in size of six Smarandache sequences are also adduced.

## 1 Introduction

In this paper some properties of six different Smarandache sequences of the 1st kind<sup>1</sup> are investigated. In particular, as we stated, the terms of these six sequences may be computed by means of one general recurrent expression

$$a_{\varphi(n)} = \sigma(a_n 10^{\psi(a_n)} + a_n + 1), \quad (1)$$

where  $a_n$  —  $n$ -th number of Smarandache sequence;  $\varphi(n)$  and  $\psi(a_n)$  — some functions;  $\sigma$  — an operator. For each of six Smarandache sequences, determined by (1), we adduce (see Sect. 2 and 3)

- a) several first numbers of the sequence;
- b) the concrete form of the analytical formula (1);
- c) the analytical formula for the calculation of  $n$ -th number in the sequence;
- d) a set of analytical formulae for constructing Magic squares  $3 \times 3$  in size from  $k$ -truncated Smarandache numbers;
- e) a few of concrete examples of Magic squares  $3 \times 3$  in size from  $k$ -truncated Smarandache numbers.

## 2 Analytical formulae yielding Smarandache sequences

1. *Smarandache numbers of  $S_1$ -series.* If  $\varphi(n) = n+1$ ,  $\sigma = 1$  and  $\psi(a_n) = [\lg(n+1)]+1$  then of (1) the following series of the numbers, denoted as  $S_1$ -series, is generated

$$1, 12, 123, 1234, 12345, 123456, \dots \quad (2)$$

The each number of

$$\chi_k = -1 + \sum_{j=0}^{[\lg(k+0,5)]} (k+1-10^j), \quad (3)$$

corresponds to each number  $a_k$  of series (2), where the notation " $[\lg(y)]$ " means integer part from decimal logarithm of  $y$ . By (3) it is easy to construct the analytical formula for the calculation of  $n$ -th number in the  $S_1$ -series:

$$a_n = 10^{\chi_n} \sum_{i=1}^n (i / 10^{\chi_i}). \quad (4)$$

By expressions

$$\Lambda^0 a_n = 1234\dots(n-1)n; \quad \Lambda^{-1} a_n = 234\dots(n-1)n; \quad \Lambda^{-2} a_n = 34\dots(n-1)n; \quad \dots \quad (5)$$

we introduce the operator  $\Lambda^{-k}$  {the operator of  $k$ -truncating the number  $a_n = 1234\dots(n-1)n$ }. Since

$$\Lambda^0 a_1 = 1, \quad \Lambda^{-1} a_2 = 2, \quad \Lambda^{-2} a_3 = 3, \quad \dots, \quad \Lambda^{-n+1} a_n = n, \quad \dots \quad (6)$$

it is evident that by the operator  $\Lambda^{-k}$  from the numbers of  $S_1$ -series one may produce the series of the natural numbers. And, vice versa, if the operator  $\Lambda^{+k}$  {the operator of  $k$ -extending the number  $n$ } is introduced:

$$\Lambda^0 n = n; \quad \Lambda^{+1} n = (n-1)n; \quad \Lambda^{+2} n = (n-2)(n-1)n; \quad \dots \quad (7)$$

then from the series of the natural numbers one may obtain the numbers of  $S_1$ -series:

$$\Lambda^0 1 = a_1, \Lambda^{+1} 2 = a_2, \Lambda^{+2} 3 = a_3, \dots, \Lambda^{n-1} n = a_n, \dots \quad (8)$$

It is evident that

a) the operators  $\Lambda^{+k}$  and  $\Lambda^{-k}$  are connected with each other. Therefore one may simplify their arbitrary combinations by the mathematical rule of the action with the power expressions {for instance,  $\Lambda^{+2}\Lambda^{-7}\Lambda^{+3} = \Lambda^{+2-7+3} = \Lambda^{-2}$ }.

b) apart from operators of  $k$ -truncating and  $k$ -extending of numbers *from the left* {see (5) and (7)} one may introduce operators of  $k$ -truncating and  $k$ -extending of numbers *from the right* {for instance,  $(\Lambda^{-2} 12345) = 345$ , but  $(12345 \Lambda^{-2}) = 123$ };

c) by means of operators of  $k$ -truncating and  $k$ -extending of numbers from the right one may represent the different relations existing between the numbers of  $S_1$ -series {for instance,  $a_n = (a_{n-1} \Lambda^{+1}) = (a_{n+1} \Lambda^{-1})$  and so on}.

2. *Smarandache numbers of  $S_2$ -series.* If  $\varphi(n) = n+1$ ;  $\sigma = \gamma$  — the operator of mirror-symmetric extending the number  $a_{[(n+1)/2]}$  of  $S_1$ -series from the right with 1-truncating the reflected number from the left, if  $n$  is the odd number, and without truncating the reflected number, if  $n$  is the even number;  $\psi(a_n) = [\lg([(n+1)/2] + 1)] + 1$ , then of (1) the following series of the numbers, denoted as  $S_2$ -series, is generated

$$1, 11, 121, 1221, 12321, 123321, 1234321, \dots \quad (9)$$

The analytical formula for the calculation of  $n$ -th number in the  $S_2$ -series has the form

$$a_n = \sum_{i=1}^{[n/2]} i 10^{\chi_i - [\lg i]} + \sum_{i=1}^{[(n+1)/2]} i 10^d, \quad (10)$$

where  $d = 1 + \chi_{[(n+1)/2]} + \chi_{[n/2]} - \chi_n$ .

3. *Smarandache numbers of  $S_3$ -series.* If  $\varphi(n) = n+1$ ;  $\sigma = \gamma$  — the operator of mirror-symmetric extending the number  $a_n$  of  $S_1$ -series from the left with 1-truncating the reflected number from the right;  $\psi(a_n) = [\lg(n+1)] + 1$ , then of (1) the following series of the numbers, denoted as  $S_3$ -series, is generated

$$1, 212, 32123, 4321234, 543212345, 65432123456, \dots \quad (11)$$

The analytical formula for the calculation of  $n$ -th number in the  $S_3$ -series has the form

$$a_n = 10^{X_n} \left\{ \sum_{i=2}^n (i \cdot 10^{X_i}) / 10^{\lceil \lg i \rceil} + \sum_{i=1}^n i / 10^{X_i} \right\}. \quad (12)$$

4. *Smarandache numbers of  $S_4$ -series.* The series of the numbers

$$1, 23, 456, 7891, 23456, 789123, 4567891, \dots \quad (13)$$

we denote as  $S_4$ -series. It is evident that the series of the numbers (13) is obtained from the infinite circular chain of the numbers

$$(123456789)(123456789) \dots (123456789) \dots \quad (14)$$

by means of the proper truncation from the left and the right. The analytical formula for the calculation of  $n$ -th number in the  $S_4$ -series has the form

$$a_n = 10^n \sum_{i=0}^{n-1} \{ 1 + d - 9 \lfloor d/9 \rfloor \} / 10^{i+1}, \quad d = i + n(n-1)/2. \quad (15)$$

5. *Smarandache numbers of  $S_5$ -series.* The series of the numbers

$$1, 12, 21, 123, 231, 312, 1234, 2341, 3412, 4123, 12345, \dots \quad (16)$$

we denote as  $S_5$ -series. By (3) it is easy to construct the analytical formula for the calculation of  $n$ -th number in the  $S_5$ -series:

$$a_n = \sum_{i=1}^z (i \cdot 10^d), \quad z = \lceil (\sqrt{8n-7} - 1)/2 \rceil, \quad (17)$$

$$d = \chi_t - \chi_i - (\chi_z + 1) \lceil (\chi_i - \chi_d) / (\chi_z + 1) \rceil, \quad t = -1 + n - z(z-1)/2.$$

6. *Smarandache numbers of  $S_6$ -series.* The series of the numbers

$$12, 1342, 135642, 13578642, 13579108642, \dots \quad (18)$$

we denote as  $S_6$ -series. The analytical formula for the calculation of  $n$ -th number in the  $S_6$ -series has the form

$$a_n = \{10^{1+2[x_{2n}/2]} \sum_{i=1}^n (2i-1) / 10^{[x_{2i-1}/2]} + \sum_{i=1}^n 2i 10^{[x_{2i}/2]} \} / 10^{[\lg 2n]}. \quad (19)$$

### 3 Magic squares $3 \times 3$ in size from $k$ -truncated Smarandache numbers

1. *Magic squares  $3 \times 3$  in size from  $k$ -truncated numbers of  $S_1$ -series.* By analysing numbers  $a_n$  of  $S_1$ -series one can conclude that it is impossible to construct an arithmetical progression from any three numbers of  $S_1$ -series. Consequently<sup>2</sup>, none Magic square  $3 \times 3$  in size can be constructed from these numbers. However, one may truncate number  $a_n$  of  $S_1$ -series from the left or/and the right by means of the operator  $\Lambda^{-k}(5)$ . Therefore there is a possibility to construct the Magic squares  $3 \times 3$  in size from truncated numbers of  $S_1$ -series. In particular, the analytical formula for constructing such Magic squares is adduced in the Fig. 1(1). If in the formula 1(1) the parameters  $n$ ,  $r$ ,  $p$  and  $q$  take, for instance, the following values:

a)  $n = 7$ ,  $r = 14$ ,  $p = 1$  and  $q = 3$ , then it generates the Magic square  $3 \times 3$  shown in the Fig. 1(2);

b)  $n = 4$ ,  $r = 0$ ,  $p = 1$  and  $q = 3$ , then the numerical square  $3 \times 3$ , shown in the Fig. 1(3), is yielded — the square 1(3) is not Magic, but it can be easy transformed to one by means of revising three numbers marked out by the dark background {the revised square see in Fig. 1(3')};

c)  $n = 4$ ,  $r = 7$ ,  $p = 1$  and  $q = 3$ , then the numerical square  $3 \times 3$ , shown in the Fig. 1(5), is yielded — the square 1(5) also is not Magic, but it can be easy transformed to one by means of revising just one number marked out by the dark background {the revised square see in Fig. 1(5')}.

By analysing the squares, shown in the Fig. 1(3) and 1(5), it can be easy understood that the analytical formula 1(1) does not hold true only in such cases when natural numbers, being components of numbers  $\Lambda^{-k}a_n$ , have different amount of digits. To obtain the Magic square in this case, one is to correct the defects of the square generated by formula 1(1) {as it made, for instance, in Fig. 1(3') and 1(5') for squares 1(3) and 1(5)}, or to change the values of parameters  $n$ ,  $r$ ,  $p$  and/or  $q$  correspondingly.

$\Lambda^{-r-p-2q} a_{n+r+p+2q}$	$\Lambda^{-r} a_{n+r}$	$\Lambda^{-r-2p-q} a_{n+r+2p+q}$
$\Lambda^{-r-2p} a_{n+r+2p}$	$\Lambda^{-r-p-q} a_{n+r+p+q}$	$\Lambda^{-r-2q} a_{n+r+2q}$
$\Lambda^{-r-q} a_{n+r+q}$	$\Lambda^{-r-2p-2q} a_{n+r+2p+2q}$	$\Lambda^{-r-p} a_{n+r+p}$

(1)

22232425262728	15161718192021	20212223242526
17181920212223	19202122232425	21222324252627
18192021222324	23242526272829	16171819202122

(2)

891011	1234	6789
3456	5678	78910
4567	9101112	2345

(3)

891011	1234	6789
3456	5678	78910
4567	101122	2345

(3')

38	10	30
18	26	34
22	42	14

(4)

15161718	891011	13141516
10111213	12131415	14151617
11121314	16171819	9101112

(5)

15161718	891011	13141516
10111213	12131415	14151617
11121314	23371819	9101112

(5')

$(r+p+2q)n + n(n+1)/2$	$rn + n(n+1)/2$	$(r+2p+q)n + n(n+1)/2$
$(r+2p)n + n(n+1)/2$	$(r+p+q)n + n(n+1)/2$	$(r+2q)n + n(n+1)/2$
$(r+q)n + n(n+1)/2$	$(r+2p+2q)n + n(n+1)/2$	$(r+p)n + n(n+1)/2$

(6)

Fig. 1. Constructing Magic squares 3x3 from  $k$ -truncated numbers of  $S_1$ -series.

It should be noted that the proper replacement of numbers  $\Lambda^{-k} a_n$  in squares 1(2), 1(3) and 1(5) by the sum of digits of natural numbers, being components of  $\Lambda^{-k} a_n$ , gives three different Magic squares  $3 \times 3$ . For instance, the Magic square, obtained by such way from square 1(3), is depicted in Fig. 1(4). The explanation of this curious fact can be found in Fig. 1(6), presenting the analytical formula of Magic square  $3 \times 3$ , which is obtained directly from the formula 1(1) by means of the mentioned way.

2. *Magic squares  $3 \times 3$  in size from  $k$ -truncated numbers of  $S_2$ -series.* To apply the methods, elaborated in point 1, for constructing Magic squares  $3 \times 3$  from numbers of  $S_2$ -series {see (9)}, we divide a set of  $S_2$ -series numbers into two different subsequences:

- 1)  $a_1=1, a_2=121, a_3=12321, a_4=1234321, \dots$
- 2)  $b_1=11, b_2=1221, b_3=123321, b_4=12344321, \dots$

By adding to the all elements of the analytical formula 1(1) from the right the operator  $\Lambda^{-k}$ , having the same form as one located from the left, we obtain the new formula of the Magic square  $3 \times 3$ . This formula allows easy to construct examples of Magic squares  $3 \times 3$  both from numbers of the first subsequence {see Fig. 2(1)} and from numbers of the second subsequence {see Fig. 2(2)}.

171819191817	101112121110	151617171615
121314141312	141516161514	161718181716
131415151413	181920201918	111213131211

(1)

17181920191817	10111213121110	15161718171615
12131415141312	14151617161514	16171819181716
13141516151413	18192021201918	11121314131211

(2)

Fig. 2. Constructing Magic squares  $3 \times 3$  from  $k$ -truncated numbers of  $S_2$ -series.



3. *Magic squares  $3 \times 3$  in size from  $k$ -truncated numbers of  $S_3$ -series.* By comparing numbers of  $S_3$ -series {see (11)} and  $S_2$ -series {see point 2} with each to other one can conclude that numbers of  $S_3$ -series resemble numbers of the first subsequence of  $S_2$ -series and distinguish from them on the order of the natural numbers movement. The example of the Magic square  $3 \times 3$  from numbers of  $S_3$ -series is presented in Fig. 3. This square is constructed by means of methods described in point 1 and 2. Thus, in spite of the mentioned difference between numbers of  $S_3$ -series and  $S_2$ -series, the methods, discussed above, can be applied for solving problems on constructing Magic square  $3 \times 3$  from numbers of  $S_3$ -series.

201918181920	131211111213	181716161718
151413131415	171615151617	191817171819
161514141516	212019192021	141312121314

Fig. 3. Constructing Magic squares  $3 \times 3$  from  $k$ -truncated numbers of  $S_3$ -series.

4. *Magic squares  $3 \times 3$  in size from  $k$ -truncated numbers of  $S_4$ -series.* In contrast to considered Smarandache sequences the digit 0 is absent in numbers of  $S_4$ -series. Besides, the order of the movement for digits 1, 2, ..., 9 can not be changed and after digit 9 can be the only digit 1. These peculiarities of numbers of  $S_4$ -series make too difficult the solving problems on constructing Magic square  $3 \times 3$ . It is evident that by using  $\Lambda^{-k}$ -operator one can easy construct classical square 4(1) {the Magic square of natural numbers from 1 to 9}. Since by means of  $\Lambda^{-k}$ -operator such square can be constructed from numbers of any Smarandache sequence {for instance, see (6)}, the example of the square 4(1) is banal. The example of the Magic square  $3 \times 3$ , presented in Fig. 4(2, 3), is less trivial.

8	1	6
3	5	7
4	9	2

(1)

$a_4 \Lambda^{-2}$	$a_1$	$\Lambda^{-1} a_3$
$a_2$	$a_3 \Lambda^{-1}$	$\Lambda^{-2} a_7 \Lambda^{-3}$
$\Lambda^{-1} a_5 \Lambda^{-2}$	$\Lambda^{-1} a_4 \Lambda^{-1}$	$\Lambda^{-3} a_6 \Lambda^{-1}$

(2)

78	1	56
23	45	67
34	89	12

(3)

Fig. 4. Constructing Magic squares  $3 \times 3$  from  $k$ -truncated numbers of  $S_4$ -series.

5. *Magic squares 3×3 in size from k-truncated numbers of  $S_5$ -series.* As compared with another Smarandache sequences of the 1st kind the numbers of  $S_5$ -series {see (16)} have the following peculiarity: the circular permutation of natural numbers is allowed in them. The analytical formula of Magic square 3×3, presented in Fig. 5(1), is just constructed with taking into account the pointed peculiarity of discussed numbers. Examples of the Magic square 3×3, obtained from formula 5(1) at  $n = 2, 3$  and 4, are depicted in Fig. 5(2, 3, 4) correspondingly. By analysing these squares it is easy to find more simple form of the analytical formula 5(1) {see Fig. 5(5), where  $a_{n-1}$  is the  $(n-1)$ th number of  $S_1$ -series,  $M$  is general amount of digits in the number  $a_{n-1}$ }.

$\Lambda^{-7} a_{n(n+15)/2 + 21}$	$\Lambda^{-5} a_{n(n+1)/2}$	$\Lambda^{-5} a_{n(n+11)/2 + 10}$
$\Lambda^{-2} a_{n(n+5)/2 + 1}$	$\Lambda^{-4} a_{n(n+9)/2 + 6}$	$\Lambda^{-6} a_{n(n+13)/2 + 15}$
$\Lambda^{-3} a_{n(n+7)/2 + 3}$	$\Lambda^{-8} a_{n(n+17)/2 + 28}$	$\Lambda^{-1} a_{n(n+3)/2}$

(1)

91	21	71
41	61	81
51	101	31

(2)

1012	312	812
512	712	912
612	1112	412

(3)

11123	4123	9123
6123	8123	10123
7123	12123	5123

(4)

$(n+7)10^M + a_{n-1}$	$n10^M + a_{n-1}$	$(n+5)10^M + a_{n-1}$
$(n+2)10^M + a_{n-1}$	$(n+4)10^M + a_{n-1}$	$(n+6)10^M + a_{n-1}$
$(n+3)10^M + a_{n-1}$	$(n+8)10^M + a_{n-1}$	$(n+1)10^M + a_{n-1}$

(5)

Fig. 5. Constructing Magic squares 3×3 from k-truncated numbers of  $S_5$ -series.

6. *Magic squares  $3 \times 3$  in size from  $k$ -truncated numbers of  $S_6$ -series.* Numbers of  $S_6$ -series {see (18)} resemble both numbers of the first subsequence of  $S_2$ -series and numbers of  $S_3$ -series {see points 2 and 3}. The example of the Magic square  $3 \times 3$  from numbers of  $S_6$ -series is presented in Fig. 6. This square is constructed by means of methods described in points 1 – 3. Thus, in spite of the mentioned difference between numbers of  $S_6$ -series and  $S_2$ -,  $S_3$ -series for solving problems on constructing Magic square  $3 \times 3$  from numbers of  $S_6$ -series the methods, discussed above, can be applied.

2527293132302826	1113151718161412	2123252728262422
1517192122201816	1921232526242220	2325272930282624
1719212324222018	2729313334323028	1315171920181614

Fig. 6. Constructing Magic squares  $3 \times 3$  from  $k$ -truncated numbers of  $S_6$ -series.

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## Perfect Powers in Smarandache Type Expressions

Florian Luca

In [2] and [3] the authors ask how many primes are of the Smarandache form (see [10])  $x^y + y^x$ , where  $\gcd(x, y) = 1$  and  $x, y \geq 2$ . In [6] the author showed that there are only finitely many numbers of the above form which are products of factorials.

In this article we propose the following

**Conjecture 1.** *Let  $a, b$ , and  $c$  be three integers with  $ab \neq 0$ . Then the equation*

$$ax^y + by^x = cz^n \quad \text{with } x, y, n \geq 2, \text{ and } \gcd(x, y) = 1, \quad (1)$$

*has finitely many solutions  $(x, y, z, n)$ .*

We announce the following result:

**Theorem 1.** *The "abc Conjecture" implies Conjecture 1.*

The proof of Theorem 1 is based on an idea of Lang (see [5]).

For any integer  $k$  let  $P(k)$  be the largest prime number dividing  $k$  with the convention that  $P(0) = P(\pm 1) = 1$ . We have the following result.

**Theorem 2.** *Let  $a, b$ , and  $c$  be three integers with  $ab \neq 0$ . Let  $P > 0$  be a fixed positive integer. Then the equation*

$$ax^y + by^x = cz^n \quad \text{with } x, y, n \geq 2, \gcd(x, y) = 1, \text{ and } P(y) < P, \quad (2)$$

*has finitely many solutions  $(x, y, z, n)$ . Moreover, there exists a computable positive number  $C$  depending only on  $a, b, c$ , and  $P$  such that all the solutions of equation (2) satisfy  $\max(x, y) < C$ .*

The proof of theorem 2 uses lower bounds for linear forms in logarithms of algebraic numbers.

**Conjecture 2.** *The only solutions of the equation*

$$x^y \pm y^x = z^n \quad \text{with } x, y, n \geq 2, z > 0, \gcd(x, y) = 1, \quad (3)$$

*are  $(x, y, z, n) = (3, 2, 1, n)$ .*

We have the following results:

**Theorem 3.** *The equation*

$$x^y \pm y^x = z^2 \quad \text{with } x, y \geq 2, \text{ and } \gcd(x, y) = 1, \quad (4)$$

*has finitely many solutions  $(x, y, z)$  with  $2 \mid xy$ . Moreover, all such solutions satisfy  $\max(x, y) < 3 \cdot 10^{143}$ .*

The proof of Theorem 3 uses lower bounds for linear forms in logarithms of algebraic numbers.

**Theorem 4.** *The equation*

$$2^y + y^2 = z^n \quad (5)$$

*has no solutions  $(y, z, n)$  such that  $y$  is odd and  $n > 1$ .*

The proof of theorem 4 is elementary and uses the fact that  $\mathbb{Z}[i\sqrt{2}]$  is an UFD.

## 2. Preliminary Results

We begin by stating the *abc Conjecture* as it appears in [5]. Let  $k$  be a nonzero integer. Define the *radical* of  $k$  to be

$$N_0(k) = \prod_{p \mid k} p \quad (6)$$

i.e. the product of the distinct primes dividing  $k$ . Notice that if  $x$  and  $y$  are integers, then

$$N_0(xy) \leq N_0(x)N_0(y),$$

and if  $\gcd(x, y) = 1$ , then

$$N_0(xy) = N_0(x)N_0(y).$$

The *abc Conjecture* ([5]). *Given  $\epsilon > 0$  there exists a number  $C(\epsilon)$  having the following property. For any nonzero relatively prime integers  $a, b, c$  such that  $a + b = c$  we have*

$$\max(|a|, |b|, |c|) < C(\epsilon)N_0(abc)^{1+\epsilon}.$$

The proofs of theorems 2 and 3 use estimations of linear forms in logarithms of algebraic numbers.

Suppose that  $\zeta_1, \dots, \zeta_l$  are algebraic numbers, not 0 or 1, of heights not exceeding  $A_1, \dots, A_l$ , respectively. We assume  $A_m \geq e^\epsilon$  for  $m = 1, \dots, l$ . Put  $\Omega = \log A_1 \dots \log A_l$ . Let  $F = \mathbb{Q}[\zeta_1, \dots, \zeta_l]$ . Let  $n_1, \dots, n_l$  be integers, not all 0, and let  $B \geq \max |n_m|$ . We assume  $B \geq e^2$ . The following result is due to Baker and Wüstholz.

**Theorem BW** ([1]). *If  $\zeta_1^{n_1} \dots \zeta_l^{n_l} \neq 1$ , then*

$$|\zeta_1^{n_1} \dots \zeta_l^{n_l} - 1| > \frac{1}{2} \exp(-(16(l+1)d_F)^{2(l+3)} \Omega \log B). \quad (7)$$

In fact, Baker and Wüstholz showed that if  $\log \zeta_1, \dots, \log \zeta_l$  are any fixed values of the logarithms, and  $\Lambda = n_1 \log \zeta_1 + \dots + n_l \log \zeta_l \neq 0$ , then

$$\log |\Lambda| > -(16ld_F)^{2(l+2)} \Omega \log B. \quad (8)$$

Now (7) follows easily from (8) via an argument similar to the one used by Shorey *et al.* in their paper [9].

We also need the following  $p$ -adic analogue of theorem BW which is due to Alf van der Poorten.

**Theorem vdP ([7]).** *Let  $\pi$  be a prime ideal of  $F$  lying above a prime integer  $p$ . Then,*

$$\text{ord}_\pi(\zeta_1^{n_1} \dots \zeta_l^{n_l} - 1) < (16(l+1)d_F)^{12(l+1)} \frac{p^{d_F}}{\log p} \Omega(\log B)^2. \quad (9)$$

We also need the following two results.

**Theorem K ([4]).** *Let  $A$  and  $B$  be nonzero rational integers. Let  $m \geq 2$  and  $n \geq 2$  with  $mn \geq 6$  be rational integers. For any two integers  $x$  and  $y$  let  $X = \max(|x|, |y|)$ . Then*

$$P(Ax^m + By^n) > C(\log_2 X \log_3 X)^{1/2} \quad (10)$$

where  $C > 0$  is a computable constant depending only on  $A, B, m$  and  $n$ .

**Theorem S ([8]).** *Let  $n > 1$  and  $A, B$  be nonzero integers. For integers  $m > 3, x$  and  $y$  with  $|x| > 1, \gcd(x, y) = 1$ , and  $Ax^m + By^n \neq 0$ , we have*

$$P(Ax^m + By^n) \geq C((\log m)(\log \log m))^{1/2} \quad (11)$$

and

$$|Ax^m + By^n| \geq \exp\left(C((\log m)(\log \log m))^{1/2}\right) \quad (12)$$

where  $C > 0$  is a computable number depending only on  $A, B$  and  $n$ .

Let  $K$  be a finite extension of  $\mathbb{Q}$  of degree  $d$ , and let  $\mathcal{O}_K$  be the ring of algebraic integers inside  $K$ . For any element  $\gamma \in \mathcal{O}_K$ , let  $[\gamma]$  be the ideal generated by  $\gamma$  in  $\mathcal{O}_K$ . For any ideal  $I$  in  $\mathcal{O}_K$ , let  $N(I)$  be the norm of  $I$ . Let  $\pi_1, \pi_2, \dots, \pi_l$  be a set of prime ideals in  $\mathcal{O}_K$ . Put

$$p = \max P(N(\pi_i)).$$

Write

$$\pi_i^h = [p_i] \quad \text{for } i = 1, \dots, l$$

where  $p_1, p_2, \dots, p_l \in \mathcal{O}_K$  and  $h$  is the class number of  $K$ . Denote by  $\mathcal{S}$  the set of all elements  $\alpha$  of  $\mathcal{O}_K$  such that  $[\alpha]$  is exclusively composed of prime ideals  $\pi_1, \pi_2, \dots, \pi_l$ . Then we have

**Lemma T.** ([9]). *Let  $\alpha \in S$ . Assume that*

$$[\alpha] = \pi_1^{b_1} \pi_2^{b_2} \dots \pi_l^{b_l}.$$

*There exist a  $\beta \in \mathcal{O}_K$  with  $|N(\beta)| \leq p^{dhl}$  and a unit  $\epsilon \in \mathcal{O}_K$  such that*

$$\alpha = \epsilon \beta p_1^{\alpha_1} p_2^{\alpha_2} \dots p_l^{\alpha_l}.$$

*Moreover,*

$$b_i = a_i h + c_i \quad \text{for some } 0 \leq c_i < h.$$

### 3. The Proofs

**The Proof of Theorem 1.** We may assume that  $\gcd(a, b, c) = 1$ . By  $C_1, C_2, \dots$ , we shall denote computable positive numbers depending only on  $a, b, c$ . Let  $(x, y, z, n)$  be a solution of (1). Assume that  $x > y$ , and that  $x > 3$ . Let  $d = \gcd(ax^y, by^x)$ . Notice that  $d \mid ab$ . Equation (1) becomes

$$\frac{ax^y}{d} + \frac{by^x}{d} = \frac{cz^n}{d}. \quad (13)$$

By the abc Conjecture for  $\epsilon = 2/3$  it follows that

$$\max(|ax^y|, |by^x|, |cz^n|) < \frac{C(2/3)N_0(abc)^{5/3}}{d^2} N_0(xyz)^{5/3}. \quad (14)$$

Let

$$C_1 = C(2/3)N_0(abc)^{5/3}$$

Since  $d \geq 1$ , and  $|b| \geq 1$ , from inequality (14) it follows that

$$y^x \leq |by^x| < C_1(xy|z|)^{5/3} < C_1 x^{10/3} |z|^{5/3}. \quad (15)$$

Since  $x > \min(y, 3)$ , it follows easily that  $y^x > x^y$ . Hence,

$$|z|^n = \left| \frac{a}{c} x^y + \frac{b}{c} y^x \right| < C_2 y^x$$

where  $C_2 = \frac{|a| + |b|}{|c|}$ . We conclude that

$$|z| < C_2^{1/n} y^{x/n} \leq C_2^{1/2} y^{x/n}. \quad (16)$$

Combining inequalities (15) and (16) it follows that

$$y^x < C_1 C_2^{5/6} x^{10/3} y^{(5x/3n)},$$

or

$$y^{x(1-5/3n)} < C_3 x^{10/3}, \quad (17)$$

where  $C_3 = C_1 C_2^{5/6}$ . Since  $2 \leq y$  and  $2 \leq n$ , it follows that

$$2^{x/6} \leq 2^{x(1-5/3n)} < C_3 x^{10/3}. \quad (18)$$

Inequality (18) clearly shows that  $x < C_4$ .

**The Proof of Theorem 2.** We may assume that

$$P \geq \max (P(a), P(b), P(c)).$$

By  $C_1, C_2, \dots$ , we shall denote computable positive numbers depending only on  $a, b, c, P$ . We begin by showing that  $n$  is bounded. Fix  $d \in \{2, 3, \dots, P-1\}$ . Suppose that  $x, y, z, n$  is a solution of (2) with  $n > 3$  and  $d \mid y$ . Since

$$by^x = cz^n - a(x^{y/d})^d \quad (19)$$

it follows, by Theorem S, that

$$P = P(by^x) = P(cz^n - a(x^{y/d})^d) > C_1 ((\log n)(\log \log n))^{1/2} \quad (20)$$

where  $C_1$  is a computable number depending only on  $a, c, d$ . Inequality (20) shows that  $n < C_2$ .

Suppose now that  $ny \geq 6$ . Let  $X = \max (x, |z|)$ . From equation (19) and theorem K, it follows that

$$P = P(by^x) = P(cz^n - ax^y) > C_3 (\log_2 X \log_3 X)^{1/2}, \quad (21)$$

where  $C_3 > 0$  is a computable constant depending only on  $a, c$ , and  $C_2$ . From inequality (21) it follows that  $X < C_3$ . Let  $C_4 = \max (C_2, C_3)$ . It follows that, if  $ny \geq 6$ , then  $\max (x, |z|, n) < C_4$ . We now show that  $y$  is bounded as well. Suppose that  $y > \max (C_4, e^2)$ . Rewrite equation (2) as

$$\frac{|cz|^n}{|a|x^y} = \left| 1 - \left( \frac{-b}{a} \right) y^x x^{-y} \right|. \quad (22)$$

Let  $A > e^e$  be an upper bound for the height of  $-b/a$  and  $C_4$ . Let  $\Omega = (\log A)^3$ . From theorem BW we conclude that

$$\log |c| + n \log |z| - \log |a| - y \log x > -\log 2 - 64^{12} \Omega \log y. \quad (23)$$

Since  $x \geq 2$ , and  $\max (x, |z|, n) < C_4$ , it follows, by inequality (23), that

$$y \log 2 - 64^{12} \Omega \log y \leq y \log x - 64^{12} \Omega \log y < C_4 \log C_4 - \log |a| + \log |c| + \log 2. \quad (24)$$

From equation (24) it follows that  $y < C_5$ .



Suppose now that  $n = y = 2$ . We first bound  $z$  in terms of  $x$ . Rewrite equation (2) as

$$z^2 = 2^x \left| \frac{b}{c} \right| \cdot \left| 1 + \left( \frac{a}{b} \right) \left( \frac{x^2}{2^x} \right) \right|. \quad (25)$$

Let  $C_6 > 0$  be a computable positive number depending only on  $a$  and  $b$  such that

$$\left| \frac{a}{b} \right| \left( \frac{x^2}{2^x} \right) < \frac{1}{2} \quad \text{for } x > C_6. \quad (26)$$

From equation (25) and inequality (26), it follows that

$$2^x \left| \frac{b}{2c} \right| < 2^x \left| \frac{b}{c} \right| \left( 1 - \left| \frac{a}{b} \right| \left( \frac{x^2}{2^x} \right) \right) < z^2 < 2^x \left| \frac{b}{c} \right| \left( 1 + \left| \frac{a}{b} \right| \left( \frac{x^2}{2^x} \right) \right) < 2^x \left| \frac{3b}{2c} \right| \quad (27)$$

for  $x > C_6$ . Taking logarithms in inequality (27) we obtain

$$xC_7 + C_8 < \log z < xC_7 + C_9 \quad \text{for } x > C_6 \quad (28)$$

where  $C_7 = \frac{\log 2}{2}$ ,  $C_8 = \frac{\log |b| - \log |2c|}{2}$ , and  $C_9 = \frac{\log |3b| - \log |2c|}{2}$ . We now rewrite equation (2) as

$$(cz)^2 - acx^2 = ab2^x. \quad (29)$$

Let  $\alpha = \sqrt{ac}$ . Then

$$(cz + \alpha x)(cz - \alpha x) = cb2^x. \quad (30)$$

We distinguish 2 cases.

CASE 1.  $ac < 0$ . Let  $K = \mathbb{Q}[\alpha]$ . Since  $ac < 0$ , it follows that all the units of  $\mathcal{O}_K$  are roots of unity. Since  $K$  is a quadratic field, it follows that the ideal  $[2]$  has at most two prime divisors. Since

$$\gcd([cz + \alpha x], [cz - \alpha x]) \mid 2[\alpha bc]$$

it follows, by lemma T, that

$$cz + \alpha x = \epsilon \beta p^u \quad (31)$$

where  $\frac{x}{h} - 1 < u \leq \frac{x}{h}$ , and  $\epsilon, \beta, p \in \mathcal{O}_K$  are such that  $|\epsilon| = 1$ ,  $|p| = 2^{h/2}$ , where  $h$  is the class number of  $K$ , and  $|\beta| < C_{10}$  where  $C_{10}$  is a computable number depending only on  $a, b$ , and  $c$ . Conjugating equation (31) we get

$$cz - \alpha x = \bar{\epsilon} \bar{\beta} \bar{p}^u. \quad (32)$$

From equations (31) and (32) it follows that

$$2\alpha x = \epsilon \beta p^u (1 - (-\epsilon^{-2})(\beta)^{-1} \bar{\beta}(\bar{p})^{-u} (\bar{p})^u).$$

Hence,

$$2|\alpha|x = |\beta||p|^u |1 - (-\epsilon^{-2})(\beta)^{-1}\bar{\beta}(p)^{-u}(\bar{p})^u| \quad (33)$$

Taking logarithms in equation (33) we obtain

$$\log(2|\alpha|) + \log x = \log |\beta| + u \log p + \log |1 - (-\epsilon^{-2})(\beta)^{-1}\bar{\beta}(p)^{-u}(\bar{p})^u|. \quad (34)$$

Let  $A$ , and  $P$  be upper bounds for the heights of  $-\epsilon^{-2}(\beta)^{-1}\bar{\beta}$  and  $p$ , respectively. Assume that  $\min(A, P) > e^e$ . Let  $\Omega = \log A(\log P)^2$ . Assume also that  $\frac{x}{h} > 1 + e^2$ . From equation (34), theorem BW, the fact that  $|p| = 2^{h/2}$ , and the fact that  $\frac{x}{h} - 1 < u \leq \frac{x}{h}$ , we obtain that

$$\begin{aligned} \log(2|\alpha|) + \log x &> \log |\beta| + u \log |p| - \log 2 - 64^{12}\Omega \log u > \\ \log |\beta| + \left(\frac{x}{h} - 1\right) \cdot \left(\frac{h}{2}\right) \log 2 - \log 2 - 64^{12}\Omega \log(x/h). \end{aligned} \quad (35)$$

Inequality (35) clearly shows that  $x < C_{11}$ .

CASE 2.  $ac > 0$ . We may assume that both  $a$  and  $c$  are positive. If  $b < 0$ , equation (2) can be rewritten as

$$|a|x^2 - |b|2^x = |c|z^2 > 0 \quad (36)$$

Equation (36) clearly shows that  $x < C_{12}$ . Hence, we assume that  $b > 0$ . We distinguish two subcases.

CASE 2.1.  $\sqrt{ac} \in \mathbb{Z}$ . In this case, from equation

$$(c|z| + \alpha x)(c|z| - \alpha x) = bc2^x$$

and from the fact that

$$\gcd(c|z| + \alpha x, c|z| - \alpha x) \mid 2\alpha cb \quad (37)$$

it follows easily that

$$\begin{cases} c|z| + \alpha x = \beta 2^u \\ c|z| - \alpha x = \gamma \end{cases} \quad (38)$$

where  $\beta, \gamma, u$  are positive integers with  $0 < \beta < bc$ ,  $\gamma < (bc) \cdot (2\alpha cb)$  and  $u > x - \text{ord}_2(2\alpha cb)$ . From equation (38) it follows that

$$2\alpha x = \beta 2^u - \gamma. \quad (39)$$

From equation (39), and from the fact that  $0 < \beta < bc$ ,  $\gamma < (bc) \cdot (2\alpha cb)$ , and  $u > x - \text{ord}_2(2\alpha cb)$ , it follows that  $x < C_{13}$ .

CASE 2.2.  $\sqrt{ac} \notin \mathbb{Z}$ . Let  $K = \mathbb{Q}[\alpha]$ . Let  $\epsilon$  be a generator of the torsion free subgroup of the units group of  $\mathcal{O}_K$ . From equation (37) and lemma T, it follows that

$$c|z| + \alpha x = \epsilon^m \beta_1 p_1^u \quad (40)$$

where  $\frac{x}{h} - 1 < u \leq \frac{x}{h}$ , and  $\beta, p_1 \in \mathcal{O}_K$  are such that  $1 < \beta_1 < C_{14}$  for some computable constant  $C_{14}$ , and  $1 < p_1 < 2^h \cdot \epsilon$ . From equation (40), it follows that

$$c|z| - \alpha x = \epsilon^{-m} \beta_2 p_2^u \quad (41)$$

where  $\beta_2 = |\beta_1|^2 / \beta_1$ , and  $p_2 = 2^h / p_1$ . Suppose now that  $x > C_6$ . Since

$$\epsilon^m = p_1^{-u} \beta_1^{-1} (c|z| + \alpha x)$$

it follows, from inequality (28), and from the fact that  $\frac{x}{h} - 1 < u \leq \frac{x}{h}$  and  $1 < p_1 < 2^h \cdot \epsilon$ , that

$$|m| < C_{15}x + C_{16} \quad \text{for } x > C_6, \quad (42)$$

for some computable constants  $C_{15}$  and  $C_{16}$  depending only on  $a, b$ , and  $c$ . From equations (40) and (41), it follows that

$$2\alpha x = \epsilon^m \beta_1 p_1^u \cdot \left(1 - \epsilon^{-2m} (\beta_1)^{-1} \beta_2 (p_1)^{-u} p_2^u\right)$$

or

$$2\alpha x = (c|z| + \alpha x) \cdot \left(1 - \epsilon^{-2m} (\beta_1)^{-1} \beta_2 (p_1)^{-u} p_2^u\right). \quad (43)$$

Let  $A_1, A_2, A_3, A_4$  be upper bounds for the heights of  $\epsilon, (\beta_1)^{-1} \beta_2, p_1, p_2$  respectively. Assume that  $\min(A_1, A_2, A_3, A_4) > e^e$ . Denote  $\Omega = \prod_{i=1}^4 \log A_i$ . Denote  $C_{17} = \max(2C_{15}, 1/h)$ . From inequality (42), it follows that

$$\max(2|m|, u) < C_{17}x + C_{16}. \quad (44)$$

Let  $B = C_{17}x + C_{16}$ . Taking logarithms in equation (43), and applying theorem BW, we obtain

$$\log(2\alpha) + \log x = \log(c|z| + \alpha x) + \log \left| 1 - \epsilon^{-2m} (\beta_1)^{-1} \beta_2 (p_1)^{-u} p_2^u \right| >$$

$$\log(c|z| + \alpha x) - \log 2 - 80^{14} \Omega \log(C_{17}x + C_{16}). \quad (45)$$

Combining inequalities (28) and (45) we obtain

$$\log(4\alpha) + \log x + 80^{14} \Omega \log(C_{17}x + C_{16}) > \log(c|z| + \alpha x) > \log z > C_7x + C_8$$

This last inequality clearly shows that  $x < C_{18}$ .

**The Proof of Theorem 3.** We treat only the equation

$$x^y + y^x = z^2.$$

We may assume that  $x$  is even. First notice that, since  $\gcd(x, y) = 1$ , it follows that  $\gcd(x, z) = \gcd(y, z) = 1$ . Rewrite equation (4) as

$$x^y = (z + y^{x/2})(z - y^{x/2}).$$

Since  $\gcd(z, y^{x/2}) = 1$  and both  $z$  and  $y$  are odd, it follows that

$$\gcd(z + y^{x/2}, z - y^{x/2}) = 2.$$

Write  $x = 2d_1d_2$  such that either one of the following holds

$$\begin{cases} z + y^{x/2} = 2^{y-1}d_1^y \\ z - y^{x/2} = 2d_2^y \end{cases} \quad \text{or} \quad \begin{cases} z + y^{x/2} = 2d_1^y \\ z - y^{x/2} = 2^{y-1}d_2^y \end{cases} \quad (46)$$

Hence, either

$$y^{x/2} = 2^{y-2}d_1^y - d_2^y \quad (47)$$

or

$$y^{x/2} = d_1^y - 2^{y-2}d_2^y \quad (48)$$

We proceed in several steps.

**Step 1.** (1) If  $x > y$  then either  $y \leq 9$  and  $x < 27$ , or  $y > 9$  and  $x < 3y$ .

(2) If  $x < y$  and  $y > 2.6 \cdot 10^{21}$ , then  $y < 4x$ .

(1) Assume first that  $x > y$ . Since

$$y^{x/2} = 2^{y-2}d_1^y - d_2^y \quad \text{or} \quad y^{x/2} = d_1^y - 2^{y-2}d_2^y$$

it follows that

$$y^{x/2} < 2^{y-1}d_1^y < (2d_1)^y < x^y \quad \text{or} \quad y^{x/2} < d_1^y < x^y. \quad (49)$$

Hence,

$$\frac{x}{2} \log y < y \log x. \quad (50)$$

Inequality (50) is equivalent to

$$\frac{x}{\log x} < 2 \frac{y}{\log y}. \quad (51)$$

If  $y \leq 9$ , then one can check easily that (51) implies  $x < 27$ . Suppose now that  $y > 9$ . We show that inequality (51) implies  $x < 3y$ . Indeed, assume that  $x \geq 3y$ . Then

$$\frac{3y}{\log 3 + \log y} = \frac{3y}{\log(3y)} \leq \frac{x}{\log x} < \frac{2y}{\log y}. \quad (52)$$

Inequality (52) is equivalent to

$$3 \log y < \log 9 + 2 \log y$$

or  $y < 9$ . This contradiction shows that  $x < 3y$  for  $y > 9$ .

(2) Assume now that  $x < y$ . Suppose first that

$$y^{x/2} = 2^{y-2} d_1^y - d_2^y.$$

In this case

$$(2d_1)^y > 2^{y-2} d_1^y = d_2^y + y^{x/2} > d_2^y$$

therefore  $2d_1 > d_2$ . Since  $x = 2d_1 d_2$ , it follows that  $2d_1 > \sqrt{x}$ , or  $d_1 > \frac{\sqrt{x}}{2}$ .

Suppose now that

$$y^{x/2} = d_1^y - 2^{y-2} d_2^y.$$

In this case,

$$d_1^y > 2^{y-2} d_2^y > d_2^y$$

or  $d_1 > d_2$ . We obtain that  $d_1 > \sqrt{\frac{x}{2}} = \frac{\sqrt{x}}{2}$ .

If equality (47) holds, it follows that

$$y^{x/2} = 2^{y-2} d_1^y \left| 1 - 2^{-(y-2)} \left( \frac{d_2}{d_1} \right)^y \right| \geq d_1^y \left| 1 - 2^{-(y-2)} \left( \frac{d_2}{d_1} \right)^y \right|. \quad (53)$$

On the other hand, if equality (48) holds, then

$$y^{x/2} = d_1^y \left| 1 - 2^{y-2} \left( \frac{d_2}{d_1} \right)^y \right|. \quad (54)$$

From inequality (53) and equation (54), we conclude that, in either case,

$$y^{x/2} \geq d_1^y \left| 1 - 2^{\epsilon(y-2)} \left( \frac{d_2}{d_1} \right)^y \right| \quad (55)$$

for some  $\epsilon \in \{\pm 1\}$ . Suppose now that  $x > e^e$ . By theorem BW, and inequality (55), it follows that

$$\frac{x}{2} \log y \geq y \log d_1 - \log 2 - 48^{10} e \log x \log y \geq$$

$$y \log \frac{\sqrt{x}}{2} - \log 2 - 48^{10} e \log x \log y \quad (56)$$

or

$$48^{10} e \log x \log y + \log 2 + \frac{x}{2} \log y > y \log \frac{\sqrt{x}}{2}. \quad (57)$$

CASE 1. Assume that  $x < 2^6$ . From inequality (57), it follows that

$$48^{10}e \cdot 6 \log 2 \cdot \log y + \log 2 + 2^5 \log y > y \log \frac{\sqrt{e^e}}{2} > \frac{y}{2}$$

or

$$(48^{10}e \cdot 6 \log 2 + 2^5) \log y + \log 2 > \frac{y}{2}$$

or

$$2(48^{10}e \cdot 6 \log 2 + 2^5 + 1) > \frac{y}{\log y}. \quad (58)$$

Let  $C_1 = 2(48^{10}e \cdot 6 \log 2 + 2^5 + 1)$ . From inequality (58) and lemma 2 in [6], it follows that

$$y < C_1 \log^2 C_1 < 2(48^{10}e \cdot 6 \log 2 + 2^5 + 1) \cdot 42^2 < 2.6 \cdot 10^{21}. \quad (59)$$

CASE 2. Assume that  $x \geq 2^6$ . Then,

$$d_1 > \frac{\sqrt{x}}{2} \geq \sqrt[3]{x}.$$

Inequality (56) becomes

$$48^{10}e \log x \log y + \log 2 + \frac{x}{2} \log y > \frac{1}{3} y \log x$$

or

$$3e48^{10} \log x \log y + \log 8 + \frac{3}{2} x \log y > y \log x$$

or

$$(3e48^{10} + 1) \log x \log y + \frac{3}{2} x \log y > y \log x$$

or

$$3e48^{10} + 1 + \frac{3}{2} \frac{x}{\log x} > \frac{y}{\log y}. \quad (60)$$

Assume first that

$$\frac{3}{2} \frac{x}{\log x} < 3e48^{10} + 1. \quad (61)$$

In this case,

$$\frac{x}{\log x} < \frac{2}{3} (3e48^{10} + 1). \quad (62)$$

Let  $C_2 = \frac{2}{3} (3e48^{10} + 1)$ . From inequality (62) and lemma 2 in [6], it follows that

$$x < C_2 \log^2 C_2 < \frac{2}{3} (3e48^{10} + 1) \cdot 41^2 < 6 \cdot 10^{20}. \quad (63)$$

In this case, from inequalities (60) and (61), it follows that

$$\frac{y}{\log y} < 2(3e48^{10} + 1). \quad (64)$$

Let  $C_3 = 2(3e48^{10} + 1)$ . It follows, by inequality (64) and lemma 2 in [6], that

$$y < C_3 \log^2 C_3 < 2(3e48^{10} + 1) \cdot 42^2 < 1.8 \cdot 10^{21}. \quad (65)$$

Assume now that  $y > 2.6 \cdot 10^{21}$ . From inequality (59), it follows that  $x \geq 2^6$ . Moreover, since inequality (65) is a consequence of inequality (61), it follows that

$$\frac{3}{2} \frac{x}{\log x} \geq 3e48^{10} + 1. \quad (66)$$

From inequalities (60) and (66) it follows that

$$\frac{3x}{\log x} > \frac{y}{\log y}. \quad (67)$$

We now show that inequality (67) implies  $y < 4x$ . Indeed, assume that  $y \geq 4x$ . Then inequality (67) implies

$$\frac{3x}{\log x} > \frac{y}{\log y} \geq \frac{4x}{\log(4x)} = \frac{4x}{\log x + \log 4}$$

or

$$3 \log x + 3 \log 4 > 4 \log x$$

or  $3 \log 4 > \log x$  which contradicts the fact that  $x \geq 2^6$ .

**Step 2.** *If  $y \geq 3 \cdot 10^{143}$ , then  $y$  is prime.*

Let

$$y^{x/2} = 2^{y-2} d_1^y - d_2^y \quad \text{or} \quad y^{x/2} = d_1^y - 2^{y-2} d_2^y. \quad (68)$$

Notice that if  $y^{x/2} = 2^{y-2} d_1^y - d_2^y$ , then  $\gcd(2d_1, d_2) = 1$ . Let  $p \mid y$  be a prime number. Since  $p \nmid 2d_1 d_2 = x$ , it follows, by theorem vdP, that

$$\frac{x}{2} \leq \max \left( \text{ord}_p(2^{y-2} d_1^y - d_2^y), \text{ord}_p(d_1^y - 2^{y-2} d_2^y) \right) < 48^{36} e \frac{p}{\log p} \log^2 y \log x. \quad (69)$$

By step 1, it follows that

$$\frac{1}{4} y < x \leq 2 \cdot 48^{36} e \frac{p}{\log p} \log^2 y \log(4y) < 4 \cdot 48^{36} e \frac{p}{\log p} \log^3 y. \quad (70)$$

Hence,

$$\frac{y}{\log^3 y} < 16 \cdot 48^{36} e \frac{p}{\log p} < 16 \cdot 48^{36} e p. \quad (71)$$

Suppose that  $y$  is not prime. Let  $p \mid y$  be a prime such that  $p \leq \sqrt{y}$ . From inequality (71) it follows that

$$\frac{\sqrt{y}}{\log^3 y} < 16 \cdot 48^{36} e$$

or

$$\frac{\sqrt{y}}{\log^3(\sqrt{y})} < 128 \cdot 48^{36} e. \quad (72)$$

Let  $k = \sqrt{y}$  and  $C_4 = 128 \cdot 48^{36} e$ . By inequality (72) and lemma 2 in [6], it follows that

$$\sqrt{y} = k < C_4 \log^4 C_4 = 128 \cdot 48^{36} e \cdot 146^4 < 5.3 \cdot 10^{71} \quad (73)$$

or

$$y < (5.3 \cdot 10^{71})^2 < 3 \cdot 10^{143} \quad (74)$$

This last inequality contradicts the assumption that  $y \geq 3 \cdot 10^{143}$ .

**Step 3.** *If  $y \geq 3 \cdot 10^{143}$ , then  $x > y$ .*

Let  $y = p$  be a prime. If  $y^{x/2} = 2^{y-2} d_1^y - d_2^y$ , it follows, by Fermat's little theorem that

$$2^{-1} d_1 - d_2 \equiv 2^{y-2} d_1^y - d_2^y \equiv y^{x/2} \equiv 0 \pmod{p},$$

therefore

$$d_1 \equiv 2d_2 \pmod{p}. \quad (75)$$

On the other hand, if  $y^{x/2} = d_1^y - 2^{y-2} d_2^y$ , then

$$d_1 - 2^{-1} d_2 \equiv d_1^y - 2^{y-2} d_2^y \equiv y^{x/2} \equiv 0 \pmod{p},$$

therefore

$$d_2 \equiv 2d_1 \pmod{p}. \quad (76)$$

Suppose that  $x < y$ . From congruences (75) and (76), we conclude that, in both cases,  $x$  is a perfect square. Hence,

$$y^x = z^2 - (\sqrt{x})^{2y} = \left(z + (\sqrt{x})^y\right) \cdot \left(z - (\sqrt{x})^y\right). \quad (77)$$

From equation (77) it follows that

$$\begin{cases} z - (\sqrt{x})^y = 1 \\ z + (\sqrt{x})^y = y^x \end{cases} \quad (78)$$

Hence,

$$2(\sqrt{x})^y = y^x - 1. \quad (79)$$

It follows, by equation (79) and theorem BW, that

$$\begin{aligned} 0 &= \log \left| y^x - 2(\sqrt{x})^y \right| = \log(y^x) + \log \left| 1 - 2y^{-x}(\sqrt{x})^y \right| > \\ &= x \log y - \log 2 - 64^{12} e \log^2 y \log x. \end{aligned} \quad (80)$$



From inequality (80) and Step 1 it follows that

$$\log 2 + 64^{12}e \log^3 y > x \log y > \frac{y \log y}{4}$$

or

$$4 \log 2 + 4 \cdot 64^{12}e \log^3 y > y \log y$$

or

$$(4 \cdot 64^{12}e + 1) \log^2 y > y$$

or

$$4 \cdot 64^{12}e + 1 > \frac{y}{\log^2 y}. \quad (81)$$

Let  $C_5 = 4 \cdot 64^{12}e + 1$ . By inequality (81) and lemma 2 in [6] it follows that

$$y < C_5 \log^3 C_5 < (4 \cdot 64^{12}e + 1) \cdot 53^3 < 8 \cdot 10^{27}. \quad (82)$$

The last inequality contradicts the fact that  $y \geq 3 \cdot 10^{143}$ .

**Step 4.** Suppose that  $y \geq 3 \cdot 10^{143}$ . Let  $y = p$  be a prime. Then, with the notations of Step 1, every solution of equation (4) is of one of the following forms:

- (1)  $y^{x/2} = 2^{y-2}d_1^y - d_2^y$  with  $y = p$ ,  $d_1 = 2 + p$ ,  $d_2 = 1$ ,  $x = 4 + 2p$
- (2)  $y^{x/2} = d_1^y - 2^{y-2}d_2^y$  with  $y = p$ ,  $d_1 = \frac{3p-1}{2}$ ,  $d_2 = 1$ ,  $x = 3p - 1$
- (3)  $y^{x/2} = d_1^y - 2^{y-2}d_2^y$  with  $y = p$ ,  $d_1 = \frac{p-1}{2}$ ,  $d_2 = 3$ ,  $x = 3p - 9$

We assume that  $y \geq 3 \cdot 10^{143}$ . In this case,  $y = p$  is prime, and  $x > y$ . From Step 1 we conclude that  $x < 3y$ . Moreover, from the arguments used at Step 1 it follows that  $d_1 > \frac{\sqrt{x}}{2}$ . Since  $x = 2d_1d_2$ , it follows that

$$d_2 < \sqrt{x} < \sqrt{3y} = \sqrt{3p}.$$

By the arguments used at Step 3 we may assume that  $x$  is not a perfect square. We distinguish the following cases.

**CASE 1.**  $d_2 = 1$ . By congruences (75) and (76) it follows that  $d_1 \equiv 2 \pmod{p}$ , or  $2d_1 \equiv 1 \pmod{p}$ .

Assume that  $d_1 \equiv 2 \pmod{p}$ . Since  $x = 2d_1$ , and  $p = y < x < 3y = 3p$ , it follows that  $d_1 = 2 + p$  and  $x = 2d_1 = 4 + 2p$ .

Assume that  $2d_1 \equiv 1 \pmod{p}$ . Again, since  $x = 2d_1$ , and  $p = y < x < 3y = 3p$ , it follows that  $d_1 = \frac{3p-1}{2}$ , and  $x = 3p - 1$ .

**CASE 2.**  $d_2 = 2$ . By congruences (75) and (76) it follows that  $d_1 \equiv 4 \pmod{p}$ , or  $d_1 \equiv 1 \pmod{p}$ . One can easily check that there is no solution in this case. Indeed, if  $d_1 \equiv 4 \pmod{p}$ , it follows that  $d_1 \geq p + 4$ . Hence,  $x = 2d_1d_2 \geq 4(p + 4) > 3p = 3y$  which contradicts the fact that  $x < 3y$ .

Similar arguments can be used to show that there is no solution for which  $d_2 = 2$  and  $d_1 \equiv 1 \pmod{p}$ .

CASE 3.  $d_2 = 3$ . By congruences (75) and (76) it follows that  $d_1 \equiv 6 \pmod{p}$ , or  $2d_1 \equiv 3 \pmod{p}$ . One can easily check that there is no solution for which  $d_1 \equiv 6 \pmod{p}$ . Suppose that  $2d_1 \equiv 3 \pmod{p}$ . Since  $p = y < x < 3y = 3p$  and  $x = 2d_1d_2 = 6d_1$ , it follows easily that  $d_1 = \frac{p-3}{2}$ , and  $x = 3p - 9$ .

CASE 4.  $d_2 = k \geq 4$ .

If  $k$  is even, then, by congruences (75) and (76), it follows that  $d_1 \equiv 2k \pmod{p}$ , or  $d_1 \equiv k/2 \pmod{p}$ . Since  $x$  is not a perfect square it follows that  $d_1 \geq p + k/2$ , therefore  $x \geq 2pk + k^2 > pk \geq 4p > 3p = 3y$  contradicting the fact that  $x < 3y$ .

If  $k$  is odd, then, by congruences (75) and (76), it follows that  $d_1 \equiv 2k \pmod{p}$ , or  $2d_1 \equiv k \pmod{p}$ . We conclude that  $d_1 \geq \frac{p-k}{2}$ , therefore  $x = 2d_1d_2 \geq k(p-k)$ . Since  $k(p-k) > 3p$  for  $5 \leq k \leq \sqrt{3p}$  and  $p \geq 3 \cdot 10^{143}$ , we conclude that  $x > 3p = 3y$  contradicting again the fact that  $x < 3y$ .

Step 5. *There are no solutions of equation (2) with  $y \geq 3 \cdot 10^{142}$  and  $x$  even.*

According to Step 4 we need to treat the following cases.

CASE 1.

$$y^{x/2} = 2^{y-2}d_1^y - d_2^y \quad \text{with } y = p, d_1 = 2 + p, d_2 = 1, x = 4 + 2p. \quad (83)$$

Hence,

$$p^{2+p} = 2^{p-2}(2+p)^p - 1 > 2^{p-3}(2+p)^p. \quad (84)$$

Taking logarithms in inequality (84) we obtain

$$(2+p) \log p > (p-3) \log 2 + p \log(p+2)$$

or

$$2 \log p + p(\log p - \log(p+2)) > (p-3) \log 2. \quad (85)$$

It follows, by inequality (85), that

$$2 \log p > (p-3) \log 2$$

or

$$p \log 2 < 2 \log p + 3 \log 2 < 5 \log p. \quad (86)$$

Inequality (86) is certainly false for  $p = y \geq 3 \cdot 10^{143}$ .

CASE 2.

$$y^{x/2} = d_1^y - 2^{y-2}d_2^y \quad \text{with } y = p, d_1 = \frac{3p-1}{2}, d_2 = 1, x = 3p-1.$$

Hence,

$$p^{(3p-1)/2} = \left(\frac{3p-1}{2}\right)^p - 2^{p-2} < \left(\frac{3p-1}{2}\right)^p < \left(\frac{3p}{2}\right)^p$$

or

$$p^{(p-1)/2} < \left(\frac{3}{2}\right)^p. \quad (87)$$

Taking logarithms in inequality (87) it follows that

$$\frac{p-1}{2} \log p < p \log 1.5$$

or

$$\log p < \frac{2p}{p-1} \log 1.5 < 3 \log 1.5 < \log 1.5^3.$$

It follows that  $p < 1.5^3 < 4$  which contradicts the fact that  $p \geq 3 \cdot 10^{143}$ .

CASE 3.

$$y^{x/2} = d_1^y - 2^{y-2} d_2^y \quad \text{with } y = p, d_1 = \frac{p-1}{2}, d_2 = 3, x = 3p-9.$$

Hence,

$$p^{(3p-9)/2} = \left(\frac{p-3}{2}\right)^p - 2^{p-2} 3^p < \left(\frac{p-3}{2}\right)^p < p^p. \quad (88)$$

From inequality (88) it follows that  $\frac{3p-9}{2} < p$  or  $p < 9$  which contradicts the fact that  $p = y \geq 3 \cdot 10^{143}$ .

**The Proof of Theorem 4.** The given equation has no solution  $(y, z, n)$  with  $n > 1$  and  $y$  odd,  $y < 5$ . Assume now that  $y \geq 5$ . We may assume that  $n$  is prime. We first show that  $n$  is odd. Indeed, assume that  $(y, z)$  is a positive solution of  $y^2 + 2^y = z^2$  with both  $y$  and  $z$  odd. Then  $(z+y)(z-y) = 2^y$ . Since  $\gcd(z+y, z-y) = 2$  it follows that  $z-y = 2$  and  $z+y = 2^{y-1}$ . Hence,  $y = 2^{y-2} - 1$ . However, one can easily check that  $2^{y-2} - 1 > y$  for  $y \geq 5$ .

Assume now that  $n = p \geq 3$  is an odd prime. Write

$$(y + 2^{(y-1)/2} \cdot i\sqrt{2}) \cdot (y - 2^{(y-1)/2} \cdot i\sqrt{2}) = z^n$$

Since  $\mathbb{Z}[i\sqrt{2}]$  is euclidian and

$$\gcd(y + 2^{(y-1)/2} \cdot i\sqrt{2}, y - 2^{(y-1)/2} \cdot i\sqrt{2}) = 1$$

it follows that there exists  $a, b \in \mathbb{Z}$  such that

$$\begin{cases} y + 2^{(y-1)/2} \cdot i\sqrt{2} = (a + bi\sqrt{2})^n \\ y - 2^{(y-1)/2} \cdot i\sqrt{2} = (a - bi\sqrt{2})^n \end{cases} \quad (89)$$

From equations (89) it follows that

$$y = \frac{(a + bi\sqrt{2})^n + (a - bi\sqrt{2})^n}{2} \quad (90)$$

and

$$2^{(y-1)/2} = \frac{(a + bi\sqrt{2})^n - (a - bi\sqrt{2})^n}{2\sqrt{2}i} \quad (91)$$

From equation (90) we conclude that  $a$  is odd. From equation (91), it follows that

$$2^{(y-1)/2} = b(na^{n-1} + s),$$

where  $s$  is even. Since both  $n$  and  $a$  are odd, it follows that  $na^{n-1} + s$  is odd as well. Hence,  $b = 2^{(y-1)/2}$ . Equation (5) can now be rewritten as

$$y^2 + 2^y = z^n = \left( (a + bi\sqrt{2}) \cdot (a - bi\sqrt{2}) \right)^n = (a^2 + 2b^2)^n$$

or

$$y^2 + 2^y = (a^2 + 2^y)^n > 2^{ny} \geq 2^{3y} \quad (92)$$

Inequality (92) implies that

$$y^2 > 2^{3y} - 2^y = 2^y(2^{2y} - 1) > 2^y,$$

which is false for  $y \geq 5$ .

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# NEW SMARANDACHE SEQUENCES: THE FAMILY OF METALLIC MEANS

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## ABSTRACT

The family of Metallic Means comprises every quadratic irrational number that is the positive solution of algebraic equations of the types

$$x^2 - nx - 1 = 0 \quad \text{and} \quad x^2 - x - n = 0,$$

where  $n$  is a natural number. The most prominent member of this family is the Golden Mean, then it comes the Silver Mean, the Bronze Mean, the Nickel Mean, the Copper Mean, etc. All of them are closely related to quasi-periodic dynamics, being therefore important clues in the study of the onset to chaos. However, they also constitute the basis of musical and architectural proportions. Through the analysis of their common mathematical properties, it becomes evident that they interconnect different human fields of knowledge, in the sense defined by Florentin Smarandache ("Paradoxist Mathematics").

Keywords: continued fractions, quadratic irrationals, Fibonacci sequences, Smarandache sequences, hyperbolic map.

## 1. INTRODUCTION

Let us introduce a new family of positive quadratic irrational numbers. The family is called the "**Metallic Means Family**" (MMF). Its members have, among other common characteristics, the one of carrying the name of a metal (see [1], [2]). E.g., the most distinguished member is the well known "Golden Mean". Then, we have the Silver Mean, the Bronze Mean, the Copper Mean, the Nickel Mean and many others.

The Golden Mean has been widely utilized by a great quantity of ancient cultures as basis of proportions to compose music, to make sculptures and paintings or construct temples and palaces (in Reference [3], see the first chapter dedicated to this subject). With respect to the many relatives of the Golden Mean, a great part of them have been used by physicists in different researchs, in trying to systematize the behavior of non linear dynamical systems that suffer the transition from periodicity to quasi-periodicity. Notwithstanding, there other instances of using these relatives in quite different fields: Jay Kappraff [4] appealed to the Silver Mean to describe and explain the roman system of proportions, making use of a mathematical property of this Mean that is, as we are going to prove, common to all the members of this curious family.

Being irrational numbers the members of the MMF, in the applications to different scientific disciplines, they have to be approximated by ratios of integer numbers and the analysis of the relation between the MMF and the approximant ratios is one of the goals of this paper. A direct consequence of this study will be the possibility of interconnecting quite distinct (sometimes opposite) human fields of knowledge, in the sense defined by Florentin Smarandache ("Paradoxist Mathematics").

## 2. CONTINUED FRACTIONS EXPANSIONS

Every real number  $x$  admits a continued fraction expansion, that is, an expression of the type

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

that is written  $x = [a_0, a_1, a_2, \dots]$ . The first coefficient can be zero (in such a case the real number is between 0 and 1) but the rest of the coefficients are positive integers. This continued fraction expansion is finite if and only if  $x$  is a rational number (that is, a number of the form  $p/q$  with  $q$  different from zero and  $p, q$  natural numbers without common factors). For example,

$$\frac{18}{7} = 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3}}} = [2, 1, 1, 3].$$

If  $x$  is an irrational number, the expansion is infinite and if we take a finite number of terms like

$$\sigma_k = a_0 + \frac{1}{a_1 + \frac{1}{\dots + \frac{1}{a_k}}}$$

we get a sequence of "*rational approximants*" to the number  $x$  such that they converge to  $x$  when  $k \rightarrow \infty$ .

Some irrational numbers, like  $\pi$  and  $e$  have approximants that converge very quickly. In particular, the number  $\pi = [3, 7, 15, 1, 292, \dots]$  converges so quickly that the third rational approximant  $\sigma_3 = \frac{335}{113} = 3.1415929\dots$  has six exact decimals!



Amazingly, this result was already known by Tsu Chung Chi in China, 5th century!. Instead, the base of the napierian logarithms, the number  $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 2, 2, 8, 1, \dots]$  converges more slowly at the beginning, due to the presence of many 'ones' in its expansion. Comparatively, the quadratic irrationals converge much slower.

Similarly to the periodic decimal expansions, the "periodic" continued fractions are denoted with a line over the period and if the continued fraction expansion is of the form  $x = [a_0, \overline{a_1, \dots, a_n}]$ , we say that the continued fraction is "purely periodic". In this context, the french mathematician Joseph Louis Lagrange (1736-1813) proved that *a real number is a quadratic irrational if and only if its continued fraction expansion is periodic (not necessarily purely periodic)*. This result was improved by Evariste Galois (1811-1832) in the following form: *The continued fraction of an irrational number  $x$  is purely periodic if and only if  $x > 1$  and it is a root of a second degree equation with integer coefficients, the other root being between -1 and 0.*

### PROPERTY Nr. 1 OF THE METALLIC MEANS FAMILY

**They are all positive quadratic irrationals.**

In fact, if we take the quadratic equation

$$(2.1) \quad x^2 - nx - 1 = 0$$

where  $n$  is a natural number and solve it, we find that the positive solutions of this equation are of the form

$$x = \frac{n + \sqrt{n^2 + 4}}{2}$$

For  $n = 1$ , the result is the well known Golden Mean  $\phi = \frac{1 + \sqrt{5}}{2} = 1.618\dots$ . To find the continued fraction expansions of this quadratic irrationals, simply we take equation (2.1) and divide it by  $x$  (different from zero):

$$x = n + \frac{1}{x}$$

Then, we replace the  $x$  of the second member iteratively by  $n + 1/x$ . In this way, we get, after  $N$  iterations:

$$x = n + \frac{1}{n + \frac{1}{n + \frac{1}{\dots + \frac{1}{n + \frac{1}{x}}}}}$$

If  $N \rightarrow \infty$ , we have

$$x = n + \frac{1}{n + \frac{1}{n + \dots}} = [\overline{n}],$$

a purely periodic continued fraction expansion.

Obviously, the Golden Mean has the most simple continued fraction expansion

$$\phi = [\overline{1}].$$

For  $n = 2$ , we have the Silver Mean  $\sigma_{Ag} = 1 + \sqrt{2}$ , which continued fraction expansion is

$$\sigma_{Ag} = 2 + \frac{1}{2 + \frac{1}{2 + \dots}} = [\overline{2}].$$

For  $n = 3$ , the result is the Bronze Mean

$$\sigma_{Br} = \frac{3 + \sqrt{13}}{2} = [\overline{3}].$$

Summarizing, solving quadratic equations of the form

$$x^2 - nx - 1 = 0$$

with  $n$  natural, we obtain as positive solutions, the members of the MMF, which continued fraction expansion is purely periodic

$$x = [\overline{n}].$$

Instead, if we solve quadratic equations of the form

$$(2.2) \quad x^2 - x - n = 0,$$

with  $n$  natural, we obtain members of the MMF which continued fraction expansion is periodic, not necessarily purely periodic, e.g.

$$[m, \overline{n_1, n_2, \dots, n_n}]$$

This last subset of Metallic Means has curious mathematical properties, with reference to the frequency of apparition of the natural numbers, as well as to the length of the period or the presence of “stable cycles” (see Reference [1] for more details).

Obviously, of all these Metallic Means, the one that converges more slowly is the Golden Mean, since all the denominators are the smallest possible — ones. This fact allows us to state the following

**The Golden Mean  $\phi$  is the most irrational of all irrational numbers.**

**Note:** In the restant possible cases of quadratic equations with integer coefficients, we find the following results, looking for positive solutions

- a)  $x^2 + nx - 1 = 0$  . Same solutions as for equation (2.1), but only their decimal part.
- b)  $x^2 + nx + 1 = 0$  . There are no positive solutions.
- c)  $x^2 - nx + 1 = 0$  . The positive solutions have periodic continued fraction expansions.
- d)  $x^2 + x - n = 0$  . The positive solutions have periodic continued fraction expansions.
- e)  $x^2 + x + n = 0$  . There are no positive solutions.
- f)  $x^2 - x + n = 0$  . There are no positive solutions.

### 3. FIBONACCI SEQUENCES

The Fibonacci sequence is a sequence of natural numbers formed by taking each number equal to the sum of the two precedent terms. For this reason, this type of sequences is called a “*secondary Fibonacci sequence*”, to distinguish them from the ternary Fibonacci sequences, in which each term is a linear combination of the three precedent terms.

Beginning with  $F(0) = 1$ ;  $F(1) = 1$ , we have the following secondary Fibonacci sequence

$$(3.1) \quad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

where

$$(3.2) \quad F(n+1) = F(n) + F(n-1).$$

Secondary Fibonacci sequences can be generalized, originating what is known as “*generalized secondary Fibonacci sequences*” GSFS, like

$$a, b, pb - qa, p(pb - qa) - qb, \dots$$

that satisfy relations of the type

$$(3.3) \quad \boxed{G(n+1) = p G(n) - q G(n-1)}$$

with  $p$  and  $q$  natural numbers.

From equation (3.3), we get

$$\frac{G(n+1)}{G(n)} = p + q \frac{G(n-1)}{G(n)} = p + \frac{q}{\frac{G(n)}{G(n-1)}}$$

Taking limits in both members of this equation and assuming that  $\lim_{n \rightarrow \infty} \frac{G(n+1)}{G(n)}$  exists and is equal to a real number  $x$  -- fact that will be proved in next theorem--, we have

$$x = p + \frac{q}{x}$$

or  $x^2 - px - q = 0$ , which positive solution is

$$x = \frac{p + \sqrt{p^2 + 4q}}{2}$$

This means that

$$(3.4) \quad \boxed{\lim_{n \rightarrow \infty} \frac{G(n+1)}{G(n)} = \frac{p + \sqrt{p^2 + 4q}}{2}}$$

Now, let us prove the existence of this limit:

### Theorem

Given a generalized secondary Fibonacci sequence (GSFS)

$$a, b, pb + qa, p(pb + qa) + qb, \dots$$

such that

$$G(n+1) = p G(n) + q G(n-1)$$

with  $p, q$  natural numbers, then there exists  $\lim_{n \rightarrow \infty} \frac{G(n+1)}{G(n)}$  and is a real positive number  $\sigma$ .

Proof: To find the  $n$ th term of the GSFS, let us put

$$G(n+1) = p G(n) + qH(n)$$

$$H(n+1) = G(n)$$

and

$$\overline{G(n)} = \begin{bmatrix} G(n) \\ H(n) \end{bmatrix}; A = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}.$$

Then it is easy to prove that

$$\overline{G(n+1)} = A \cdot \overline{G(n)}.$$

Let us assume that  $G(0) = G(1) = 1$  for simplicity. If  $\overline{G(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  then  $\overline{G(n+1)} = A^n \cdot \overline{G(1)}$  and the problem is reduced to the finding of the  $n$ th power of the matrix  $A$ . We know that the eigenvalues of  $A$  are

$$\sigma = \frac{p + \sqrt{p^2 + 4q}}{2}; \sigma' = \frac{p - \sqrt{p^2 + 4q}}{2}.$$

To diagonalize  $A$  so as to transform it in  $A_d = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma' \end{bmatrix}$ , we shall use the change of base matrix  $P = \begin{bmatrix} \sigma & \sigma' \\ 1 & 1 \end{bmatrix}$ . The  $n$ th power of  $A$  is calculated applying the similarity transformation

$$A^n = P \cdot A_d^n \cdot P^{-1} = \frac{1}{\sigma - \sigma'} \begin{bmatrix} \sigma^{n+1} - \sigma'^{n+1} & \sigma\sigma'(\sigma'^n - \sigma^n) \\ \sigma^n - \sigma'^n & \sigma\sigma'(\sigma'^{n-1} - \sigma^{n-1}) \end{bmatrix}.$$

and the  $n$ th term of the GSFS

$$1, 1, p + q, p(p + q) + q, \dots$$

is given by the following expression

$$G(n+1) = \frac{\sigma^{n+2} - \sigma'^{n+2}}{\sigma - \sigma'}.$$

Replacing  $\sigma - \sigma' = \sqrt{p^2 + 4q}$ ;  $\sigma' = -\frac{q}{\sigma}$  we have

$$\lim_{n \rightarrow \infty} \frac{G(n+1)}{G(n)} = \lim_{n \rightarrow \infty} \frac{\sigma^{n+1} + \left(\frac{q}{\sigma}\right)^{n+1}}{\sigma^n + \left(\frac{q}{\sigma}\right)^n} = \sigma$$

and the proof is completed.

Note: if instead of choosing  $G(0) = G(1) = 1$  we begin the GSFS with two arbitrary values  $a$  and  $b$ , it is easy to prove that the result is the same. Indeed, given the GSFS

$$a, b, pb + qa, p(pb + qa) + qb, \dots$$

we have to evaluate the quotient

$$\frac{G(n+1)}{G(n)} = \frac{pbG(n) + qaG(n-1)}{pbG(n-1) + qaG(n-2)} = \frac{pb \frac{G(n)}{G(n-1)} + qa}{pb + \frac{qa}{\frac{G(n-1)}{G(n-2)}}} \xrightarrow{n \rightarrow \infty} \sigma.$$

Let us put  $G(0) = G(1) = 1$  and consider different possibilities for the coefficients of (3.4). Then, if  $p = q = 1$ , we have the Golden Mean

$$x = \frac{1 + \sqrt{5}}{2} = \phi = [\overline{1}].$$

If  $p = 2$  and  $q = 1$ , the sequence has the form

$$(3.5) \quad 1, 1, 3, 7, 17, 41, 99, 140, \dots$$

where

$$(3.6) \quad G(n+1) = 2G(n) + G(n-1),$$

and from (3.4) we get the Silver Mean

$$\sigma_{Ag} = \lim_{n \rightarrow \infty} \frac{G(n+1)}{G(n)} = [\overline{2}].$$

Analogously, if  $p = 3$  and  $q = 1$ , the sequence is

$$(3.7) \quad 1, 1, 4, 13, 43, 142, 469, \dots$$

where

$$(3.8) \quad G(n+1) = 3 G(n) + G(n-1),$$

and we get the Bronze Mean

$$\sigma_{Br} = \lim_{n \rightarrow \infty} \frac{G(n+1)}{G(n)} = \frac{3 + \sqrt{13}}{2} = [\bar{3}],$$

If  $p = 1$  and  $q = 2$ , the sequence is

$$(3.9) \quad 1, 1, 3, 5, 11, 21, 43, 85, \dots$$

where

$$G(n+1) = G(n) + 2 G(n-1)$$

and we get the Copper Mean

$$\sigma_{Cu} = 2 = [2, \bar{0}].$$

If  $p = 1$  and  $q = 3$ , the sequence is

$$(3.10) \quad 1, 1, 4, 7, 19, 40, 97, \dots$$

where

$$G(n+1) = G(n) + 3 G(n-1)$$

and we get the Nickel Mean

$$\sigma_{Ni} = \lim_{n \rightarrow \infty} \frac{G(n+1)}{G(n)} = \frac{1 + \sqrt{13}}{2} = [2, \bar{3}].$$

Summarizing our results, we may affirm

**PROPERTY Nr. 2 OF THE METALLIC MEANS FAMILY**

**All of them are obtained as limits of ratios of two consecutive terms of generalized secondary Fibonacci sequences.**

## 4. ADDITIVE PROPERTIES

(3.1) Let us form now the sequence of ratios of consecutive terms of the sequence

$$(4.1) \quad \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \dots$$

Obviously, this sequence converges directly to the Golden Mean  $\phi$ . This sequence is very useful as a good approximation: indeed the term  $u(11) = 233/144 = 1.6180$  with four exact decimals!

If we take now a geometric progression of ratio  $\phi$  such as

$$\dots, \frac{1}{\phi^2}, \frac{1}{\phi}, 1, \phi, \phi^2, \phi^3, \dots$$

we can easily verify that this geometric progression is also a GSFS. In fact

$$\frac{1}{\phi^2} + \frac{1}{\phi} = \frac{1+\phi}{\phi^2} = 1.$$

The same happens for the Silver Mean  $\sigma_{Ag}$ , starting from the sequence

$$(4.2) \quad \frac{1}{1}, \frac{3}{1}, \frac{7}{3}, \frac{17}{7}, \frac{41}{17}, \frac{99}{41}, \frac{140}{99}, \dots,$$

that converges to  $\sigma_{Ag}$ . The sequence

$$\dots, \frac{1}{\sigma_{Ag}^2}, \frac{1}{\sigma_{Ag}}, 1, \sigma_{Ag}, \sigma_{Ag}^2, \sigma_{Ag}^3, \dots$$

is a geometric progression of ratio  $\sigma_{Ag}$  that satisfies condition (3.6). Indeed

$$\frac{1}{\sigma_{Ag}} + 2 = \sigma_{Ag}; \quad 1 + 2\sigma_{Ag} = \sigma_{Ag}^2; \quad \sigma_{Ag} + 2\sigma_{Ag}^2 = \sigma_{Ag}^3; \dots$$

Similarly, it is easy to prove that the sequence of ratios

$$(4.3) \quad \frac{1}{1}, \frac{4}{1}, \frac{13}{4}, \frac{43}{13}, \frac{142}{43}, \frac{469}{142}, \dots,$$

converges to the Bronze Mean  $\sigma_B = \frac{3+\sqrt{13}}{2} = [\bar{3}]$  and the sequence



$$\dots, \frac{1}{\sigma_B^2}, \frac{1}{\sigma_B}, 1, \sigma_B, \sigma_B^2, \sigma_B^3, \dots$$

is a geometric progression of ratio  $\sigma_B$  that satisfies condition (3.6). This is due to the fact that

$$\frac{1}{\sigma_B} + 3 = \sigma_B; 1 + 3\sigma_B = \sigma_B^2; \sigma_B + 3\sigma_B^2 = \sigma_B^3; \dots$$

Similarly for all GSFS. These numerical sequences (4.1), (4.2), (4.3), and so on, are new Smarandache sequences that have to be empirically used as approximations to the values of the members of the MMF. Furthermore, the sequences formed by taking these members as ratios enjoy the following unique mathematical property:

**PROPERTY Nr. 3 OF THE METALLIC MEANS FAMILY**

**They are the only positive quadratic irrational numbers that originate GSFS (with additive properties) which are, simultaneously, geometric progressions.**

This curious property of satisfying both arithmetic additive and geometric properties, bestow all the members of the MMF with interesting characteristics to become basis of different systems of geometric proportions in Design.

## 5. PROPORTIONS SYSTEMS

The golden Mean  $\phi = \frac{1+\sqrt{5}}{2}$ , is indissolubly linked to pentagonal symmetry. Indeed, if we take a regular pentagon of unitary edge, like the one depicted in Fig. 5.1, it is easy to prove that its diagonal is equal to  $\phi$ . Considering the geometric similarity of the two isosceles triangles ADC and ABF we have

$$\frac{AD}{DC} = \frac{DC}{AD - FD}$$

Being  $DC = FD = 1$  and calling  $x = AD$ , we obtain the quadratic equation  $x(x - 1) = 1$  or  $x^2 - x - 1 = 0$ , that is equation (2.1) with  $n = 1$  and positive solution  $x = \phi$ . It is not difficult to prove besides the following “golden relations” in the regular pentagon

$$GB = \phi - 1 = \frac{1}{\phi} = 0,618...$$

$$GI = FG = 1 - \frac{1}{\phi}$$

$$FG = \frac{1}{\phi^2} = 0,382...$$

$$JG = \frac{1}{\phi^3} = 0,236...$$

These “golden relations” determine, for example, the proportions of the ancient mask of Hermes (Medusa), shown in Fig. 5.2. It is a wonderful Roman marble after Greek original, 1st century BC. , pertaining to the artistic collection of the Glyptothek, Munich, Germany.

Innumerable are the references to the apparition of the Golden Mean  $\phi$  in the proportion systems adopted by antique civilizations in their constructions, as well as its presence in the human body proportions and in Botany. Among the many authors that have dedicated their researchs to this subject, we have to mention Matila Ghyka [5], [6] and [7], H. E. Huntley [8] and Theodore Andrea Cook, whose book [9], published in 1979, is a reprint of the original published by Constable, London, England, as early as 1914.

Instead, the Silver Mean is linked to octogonal symmetry, as it is shown in Fig. 5.3. “Silver relations” have been found in many examples, coming from quite different fields of human knowledge. In particular, the mathematician Jay Kappraff [4], at the conference *Nexus'96: Relations between Architecture and Mathematics*, that took place in Fucecchio (province of Florence) in June 1996, carried out a carefully analysis of the three architectonic proportion systems presented by P. H. Scholfield in his excellent book [10]. These three proportion systems are the following

- 1) the system of musical proportions used during the Italian Renaissance, developed by Leon Battista Alberti [11];
- 2) the Modulor created by the twentieth-century architect Le Corbusier [12] and
- 3) the Roman proportion system.

The musical system was based on rational proportions inherent in the musical scale. Although it succeeded in creating harmonic relationships in which key proportions were repeated in a design, this system did not have the additive properties necessary for a successful proportion system. Notwithstanding, the very well known contemporary Modulor that is based on the Golden Mean  $\phi$  , and the ancient Roman proportion system, based on the Silver Mean, both conform to the relationships inherent in the system of musical proportions, with the great advantage of having additive properties.

Unlike the Renaissance system, which used a static sequence of commensurable ratios to proportion the length, width and height of rooms, Le Corbusier's system

developed a scale of lengths based on the irrational number  $\phi$ , through a GSFS and geometric sequence:

$$\dots, \frac{a}{\phi^2}, \frac{a}{\phi}, a, a\phi, a\phi^2, a\phi^3, \dots$$

for some convenient unit  $a$ , directly determined by ergonomic reasons. In general, the ratios involved in this system are incommensurable and Le Corbusier, in his designs, used an integer GSFS approximation, that is a Smarandache sequence. More details about this proportion system may be consulted in References [13] and [14].

Now, we are going to consider in detail the third proportion system. With this purpose, let us consider a couple of sequences

$$(5.1) \quad \begin{array}{cccccccc} & 1 & & 3 & & 7 & & 17 & & 41 & \dots \\ 1 & 1 & 2 & & 5 & & 12 & & 29 & & 70 & \dots \end{array}$$

such that

$$(5.2) \quad A(n+2) = 2A(n+1) + A(n).$$

These sequences satisfy three additive fundamental properties: in addition to relation (5.2) they obey the following numerical relations

$$\begin{aligned} 7 &= 2 \cdot 3 + 1; 17 = 2 \cdot 7 + 3; \dots \\ 5 &= 2 \cdot 2 + 1; 12 = 5 \cdot 2 + 1; \dots \end{aligned}$$

and

$$\begin{aligned} 2 + 5 &= 7; 5 + 12 = 17; 12 + 29 = 41; \dots \\ 2 + 3 &= 5; 5 + 7 = 12; 12 + 17 = 29; 29 + 41 = 70; \dots \end{aligned}$$

Furthermore, the ratios of diagonally adjacent terms of the sequences (5.1) are approximations to  $\sqrt{2}$

$$(5.3) \quad \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \dots \rightarrow \sqrt{2}.$$

But since the sum of any couple of numbers of the upper sequence, is not represented in this system, we may expand it adding a third sequence obtained by duplicating the terms of the lower sequence

$$(5.4) \quad \begin{array}{cccccccc} & & 2 & & 4 & & 10 & & 24 & & 58 & \dots \\ 1 & 1 & 2 & 3 & 5 & 7 & 12 & 17 & 29 & 41 & 70 & \dots \\ & 1 & 2 & 3 & 5 & 7 & 12 & 17 & 29 & 41 & 70 & \dots \end{array}$$

Finally, the Roman architectonic system utilizes the following incommensurable schema based on the Silver Mean, which is equivalent to the commensurable system (5.4)

$$\begin{array}{cccccccc}
 & & 2\sqrt{2} & & 2\sqrt{2}\sigma_{Ag} & & 2\sqrt{2}\sigma_{Ag}^2 & & 2\sqrt{2}\sigma_{Ag}^3 & & \dots \\
 (5.5) & & 2 & & 2\sigma_{Ag} & & 2\sigma_{Ag}^2 & & 2\sigma_{Ag}^3 & & \dots \\
 & \sqrt{2} & & \sqrt{2}\sigma_{Ag} & & \sqrt{2}\sigma_{Ag}^2 & & \sqrt{2}\sigma_{Ag}^3 & & \dots & \\
 1 & & \sigma_{Ag} & & \sigma_{Ag}^2 & & \sigma_{Ag}^3 & & \dots & & 
 \end{array}$$

This system holds all the additive relations of sequences (5.4), as it is easy to prove. Donald and Carol Watts [15], a couple of american architects, have carefully studied the ruins of the Garden Houses at Ostia, the city-port of the Roman Empire and they found that all these houses have been designed using theoretically the proportion system (5.5) and practically, its integer approximation (5.4). These are not the only examples of the antiquity where the Silver Mean is present, since the italian-american architect Kim Williams has found similar results while surveying:

- 1) the pavement of the baptistery of San Giovanni, Florence, Italy [16],
- 2) Verrocchio's Tomb slab for Cosimo de' Medici, patriarch of the wealthiest of Florentine families [17] and
- 3) the famous Medici Chapel in Florence, Italy, built by Michaelangelo [18].

## 6. FRACTAL STRUCTURES OF ST. GEORGE

Alan St. George is a British retired architect, living in Portugal and dedicated to the creation of mathematical sculptures. In december 1995 he presented at Lisboa his exposition "*La forma del número*" [19]. His originals are fabricated with acrylic or metallic plates and they can be reproduced by computerized graphics. The generation of these original structures is based on the fractal principle of adding to each one of the five platonic solids -- tetrahedra, cube or hexahedra, octahedra, dodecahedra, icosahedra -- reduced versions of the same solid. In such a way, adding in each iteration auto-similar versions of the original structure, the result are fractal variations of regular solids.

For example, to convert a cube in a fractal octahedra, we begin with a cube which faces are divided in nine equal squares, as indicated in Fig. 6.1. Then, we build a cross with six smaller cubes, which faces are of the size of the above mentioned squares. Five of these cubes are located in form of a "*greek cross*" and the sixth is put over the central cube, forming a sort of stepping pyramid. The construction goes on sticking one of such units over each face of the original cube. Then, each of the faces of the resulting structure is subdivided in nine even smaller squares, over which we stick more reduced copies of the stepping pyramid.

It is also possible to fractalize an octahedra and obtain a tetrahedra or a cube, like the mathematician Ian Stewart suggested in an interesting paper [20]. And why not? It would also be feasible to apply this fractalization process to semi-regular solids, a task that has not been focussed yet ...

Another variant of St. George consists in constructing three-dimensional spirals, starting also from the five platonic solids. In particular, let us consider the icosahedra of pentagonal symmetry (Fig. 6.2), which main characteristics we detail in what follows

Faces: 20      Vertices: 12      Edges: 30

Edge length: 1

Distance from the polyhedra centre to the face centre:  $\phi^2 / 2\sqrt{3} = 0,7558...$

Distance from the polyhedra centre to the edge mid-point:  $\phi / 2 = 0,8090...$

Distance from the polyhedra centre to a vertex:  $\sqrt[4]{5} \sqrt{\phi} / 2 = 0,9511...$

Volumen:  $5\phi^2 / 6 = 2,1817...$

Starting with an icosahedra, it is possible to construct the so called “*icosahedral spiral*”, following a path that passes through the twelve triangular edges of the icosahedra, visiting each vertex once and only once (Fig. 6.3). The construction is fulfilled by means of a sequence of “legs”, which correspond to the twelve edges of the icosahedra. Each leg is connected to the previous one and is parallel to an edge. But the successive legs have different lengths: each of them has  $\phi^{1/12} = 1,040916...$  times the length of its antecessor. The answer to the question: why this strange figure?, is that after having added twelve edges to a given one, the last edge is parallel to the original, having increased its length in  $(\phi^{1/12})^{12} = \phi$ .

Obviously, the choice of the Golden Mean  $\phi$  in the construction of the icosahedral spiral of St. George, obeys to mathematical as well as purely aesthetic reasons. In any case, it is impossible to deny the underlying mathematical reality inherent to a pentagonal symmetry so directly related to the Golden Mean ...

## 7. INFLATIONARY SYSTEM

We may consider that the terms of the different GSFS that define the Metallic Means family, can be ordered in generations in such a way that each generation “inherits” a property from his antecessor. This type of inheritance is completely normal in iterative processes and frequently, produces auto-similar structures that are the base of fractal configurations [20]. Let us denote such processes as “*inflationary*”, using an usual noun in Economy.

Let us consider two types of building blocks  $A$  and  $B$  that are distributed according to the inflation schema

$$S_{p+1} = S_{p-1}^m S_p^n$$

where  $m$  and  $n$  are integers;  $p \geq 2$ .  $S_L^m$  represents  $m$  adjacent repetitions of the stack  $S_L$ .

It is easily proved that the Golden Mean  $\phi$  is generated by the recurrence relation

It is easily proved that the Golden Mean  $\phi$  is generated by the recurrence relation

$$S_{p+1} = S_{p-1} S_p ,$$

that is,

$$S_1 = \{A\}; S_2 = \{BA\}; S_3 = \{ABA\}; S_4 = \{BAABA\}; \dots$$

in which each term is the “sum” of its two immediate antecessors.

The Silver Mean, instead, is generated by the recurrence relation

$$S_{p+1} = S_{p-1} S_p^2 ,$$

$$S_1 = \{A\}; S_2 = \{BA\}; S_3 = \{ABABA\}; S_4 = \{BAABABAABABA\}; \dots$$

such that each term of the chain is formed by writing contiguously two replicas of the precedent term and adding its antecessor to the left of the replicas.

In the case of the Bronze Mean, the relation is

$$S_{p+1} = S_{p-1} S_p^3 ,$$

$$S_1 = \{A\}; S_2 = \{BA\}; S_3 = \{ABABABA\}; S_4 = \{BAABABABAABABABAABABABA\}; \dots$$

For the Copper Mean, we have the relation

$$S_{p+1} = S_{p-1}^2 S_p$$

$$S_1 = \{B\}; S_2 = \{A\}; S_3 = \{BBA\}; S_4 = \{AABBA\}; \dots$$

And for the Nickel Mean

$$S_{p+1} = S_{p-1}^3 S_p$$

$$S_1 = \{B\}; S_2 = \{A\}; S_3 = \{BBBA\}; S_4 = \{AAABBBBA\}; \dots$$

Finally, we may assert

#### PROPERTY Nr. 4 OF THE METALLIC MEANS FAMILY

All the members of this family are obtained through an "inflationary schema" that produces a binary chain originated by two primitive blocks  $A$  and  $B$  that are distributed according to the inflation schema

$$S_{p+n} = S_{p-1}^m S_p^n$$

where  $m$  and  $n$  are integers and  $p \geq 2$ .

### 8. THE HYPERBOLIC MAP

In analyzing dynamical systems -- that is, physical systems which behavior changes with time -- it is crucial to detect periodic orbits. This periodic behavior, as well as the transition to quasi-periodic orbits, is mathematically studied considering irrational values of some characteristic parameter and, in such a case, as the important fact is the "irrationality" of such a value, the integer part is omitted and only the decimal part of the number is taken into account. More precisely, the main subject is restricted to the analysis of **maps** (transformations) of the unitary interval  $(0,1)$  in itself.

Returning to the continued fraction expansion, there is another possibility of expressing the continued fraction expansion of a positive real number  $\alpha < 1$ . Let us put  $x_1 = \frac{1}{\alpha}$  and apply the iterative process described by the following relation

$$(8.1) \quad x_n = \frac{.1}{\text{mant } x_{n-1}}$$

where  $\text{mant } x$  means "*mantissa of x*" and is the rest of the number  $x$  when it is taken modulo 1, that is, when one subtracts as many times 1 as possible.

E.g.  $\text{mant } \pi = 0,1416\dots$ ;  $\text{mant } \phi = 0.618\dots$

Then we may state that the continued fraction expansion of the number  $\alpha$  is

$$[\underline{x}_0], [\underline{x}_1], \dots ]$$

where  $\underline{x}_-$ , the so called "*floor function*" by Manfred Schroeder [21], is the biggest integer not greater than  $x_-$ .

Notice that:

$$\text{mant } \phi = 1/\phi$$

or

$$\text{mant } \phi = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}} = [0, 1, 1, \dots] = [0, \overline{1}].$$

The iterative process (8.1) is called the “hyperbolic map” [22]. This map is very simple to execute if the number  $x$  is given as a continued fraction expansion:

*In each iteration move all the terms of the expansion  $x = [a_0, a_1, a_2, \dots]$  one place to the left and leave out the first coefficient of the expansion.*

In Fig. 8.1a we show the iteration of the hyperbolic map, starting from the number  $x = \pi$  and in Fig. 8.1b the ordered sequence of 200 points is depicted. The same procedure have been applied to the hyperbolic map starting from the number  $e$  (see Figs. 8.2a y 8.2b). It is highly interesting to compare in both cases the graphics 8.1a and 8.1b as well as 8.2a and 8.2b: notice how the 200 points of the hyperbolic map ordered themselves when in reality, they are following a completely chaotic<sup>1</sup> [24] trajectory!

Obviously, being the continued fraction expansion of the Golden Mean a purely periodic expansion, it is a “fixed point” or an “equilibrium value” of the hyperbolic map, through all the iterations. That means that if the initial value is  $A(0) = a$ , then  $A(k) = a$  is a constant solution to the iterated dynamical system, for all values of  $k$ .

The same happens with all the members of the family that have a purely periodic continued fraction expansion. In the restant cases, where the continued fraction expansion is only periodic, we have also fixed points of the hyperbolic map, since leaving aside the first iteration, then the obtained value is invariant.

In fact, we have depicted in Fig. 8.3 the hyperbolic map starting from the Golden Mean  $\phi$  and in Fig. 8.4 the hyperbolic map starting from all the others Metallic Means we have already considered. As is easily seen, they appear as fixed points of the hyperbolic map. We have taken 50 digits and 1,000 iterations.

In conclusion, we may assert

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<sup>1</sup> “Chaotic” is a process with respect to its dynamics, that is, when it is not possible to adventure any prognosis about its future evolution, since very similar initial conditions produce behaviors of the system that differ enormously among them.



## PROPERTY NR. 5 OF THE METALLIC MEANS FAMILY

Since the continued fraction expansions of the Golden, Silver and Bronze Means are of the form  $[1]$ ,  $[2]$ ,  $[3]$ , respectively, these numbers are “fixed points” of the hyperbolic map. For the restant members of this family, that possess periodic continued fraction expansions of the form  $[\alpha, \overline{n}]$ , being all the terms (with the exception of the first) equal to  $n$ , we have also fixed points of the hyperbolic map.

NOTE: Of course, the number of members of the MMF that satisfies Properties 1, 2, 3, 4 and 5, is infinite, since we could add to the above mentioned irrational numbers, all the irrational numbers which continued fraction expansion is purely periodic of period 1, such as for example

$$[4] = 1 + 2\phi; [5] = \frac{5 + \sqrt{29}}{2}; [6] = 3 + \sqrt{10}; [7] = \frac{7 + \sqrt{53}}{2}; [8] = 4 + \sqrt{17}; \dots$$

as well as all the possible combinations of continued fraction expansions of the form  $[n, \overline{p}]$ , with  $n$  natural and  $p$  an uneven number:

$$[2, \overline{3}] = \frac{1 + \sqrt{13}}{2}; [3, \overline{5}] = \frac{1 + \sqrt{29}}{2}; [4, \overline{7}] = \frac{1 + \sqrt{53}}{2}.$$

The rest of the members of the family are integer numbers with continued fraction expansions  $[n, \overline{0}]$  or else numbers with continued fraction expansions that include “stable cycles” obeying certain regularity rules that will be published elsewhere. Some of them are

$$\frac{1 + \sqrt{21}}{2} = [2, \overline{1, 3}]; \frac{1 + \sqrt{33}}{2} = [3, \overline{2, 1, 2, 5}]; \frac{1 + \sqrt{73}}{2} = [4, \overline{1, 3, 2, 1, 1, 2, 3, 1, 7}].$$

## 9. QUASI-CRYSTALS: FORBIDDEN SYMMETRIES

Among the many problems in Physics, Chemistry, Biology and Ecology where the members of the MMF appear, one of the most striking is the structure of a quasi-crystal. The most symmetric, regular and periodic of all real entities, are the “*crystals*”. At the opposite end of the scale, we have the disordered or amorphous substances, like the “*glasses*”.

To distinguish between a crystal and a glass let us consider that a real crystal can be modelized putting an atom or a molecule at all the vertices of a regular triangular, quadrangular or hexagonal lattice, lattices that have symmetries of order 3, 4 and 6 (Fig. 9.1). In such a way, the problem of matter structure is reduced to one of pure geometry. This was the state of the art until 1984, when Schechtman et al. [25], [26], registering diffraction schema of electrons in an alloy of Aluminium and Manganese quickly cooled, found in cutting with planes forming determined angles, pentagonal symmetries of order 5, wholly impossible in a crystal since it is, obviously, impossible to tessellate the plane with regular pentagons.

These configurations with pentagonal symmetry, that possess a quasi-periodic spatial structure, were called "*quasi-crystals*". And they are really a new solid state of matter!

What is extremely interesting is the fact that the projections were taken cutting with a plane which slope with respect to the ground was equal to the Golden Mean  $\phi$ .

Starting with this discovery, there appeared another quasi-crystals with other forbidden symmetries. E.g. the Silver Mean  $\sigma_{Ag} = 1 + \sqrt{2} = [ \bar{2} ]$ , generates a quasi-crystal with a forbidden symmetry of order 8 (see [27], [28]), while  $[ \bar{4} ] = \phi^3$  appears in another forbidden symmetry, of order 12 (see [29]). Both symmetries, have been empirically detected.

In particular, Gumbs, Ali et al., in various highly interesting papers [30], [31], [32], [33] and [34] studied electronic, optical, acoustic and super-conducting properties of quasi-periodic layered systems. For that purpose, they constructed geometric one-dimensional models of a new type of quasi-crystals devised taking as basis GSFS. They were interested in these quasi-crystals because of their important physical applications, i.e. the problem of light transmission through a multi-layered medium. Among their most remarkable experimental results, they found fundamental differences in the behavior of Metallic Means which continued fraction expansion is purely periodic (the Golden Mean, the Silver Mean and the Bronze Mean) and the Metallic Means with only periodic continued fraction expansions (the Copper Mean and the Nickel Mean):

- 1) In studying the electronic properties of a GSFS lattice, it was found that the trace maps of the Golden, Silver and Bronze Mean lattices are volume-preserving (non-dissipative) while those of the Copper and Nickel Mean lattices are volume-non-preserving (dissipative).
- 2) In investigating the magnetic excitation spectra of a Nickel-Molybdenum GSFS lattice, it was found that only in the case of purely periodic continued fraction expansions, the whole spectrum is self-similar. In the case of periodic continued fraction expansions, only some parts of the whole spectrum are self-similar.

3) In considering quasi-periodic quantum Ising models in which the exchange interaction follows a GSFS, it was proved that in the case of dissipative maps (Copper and Nickel Mean lattices), the spectral properties are directly determined by the attractor of the map. And that the Copper and Nickel Mean lattices can be classified as between quasi-periodic and random, with the Nickel Mean more random than the Copper Mean.

## 10. CANTOR SPECTRA IN CRITICAL STATES

In 1919, the brilliant mathematician Félix Hausdorff published a fundamental paper on the concept of "*dimensión*" of a set. This paper opened the possibility of constructing sets with non integer topological dimension! The topological dimension corresponds to the common meaning of the word "dimension" and is an integer: it is zero for a point, one for a straight line, two for a certain portion of the plane and three for any body in space. But evidently, the curves, surfaces and volumes may be so complex as to make it necessary to differentiate among them, taking into account how quickly the length, the surface or the volumen vary with respect to measure scales each time smaller. This notion established the base to define the "**fractal dimension**", introduced by the polish mathematician Benoit B. Mandelbrot [35], [36].

Mandelbrot defined a "**fractal**" as a set with a Hausdorff dimension greater or equal to its topological dimension. It can be stated that the concept of dimension he used was a simplification of Hausdorff dimension.

The notion of self-similarity is strictly related with the intuitive concept of dimension. A segment may be divided into  $N$  equal sub-segments, each of which is in a relation  $\varepsilon = 1/N$  with the original segment (Fig. 10.1). Analogously, in dividing a square into  $N$  equal sub-squares, obviously self-similar, we have a relation  $\varepsilon = 1/N^{1/2}$  with the complete figure; this ratio is  $\varepsilon = 1/N^{1/3}$  in the case of a cube and  $\varepsilon = 1/N^D$  for a  $D$ -dimensional object. Then

$$\varepsilon^D = 1/N.$$

Taking logarithms in both members, we get

$$D \ln \varepsilon = - \ln N,$$

from where we get the fractal dimension  $D$ :

$$(10.1) \quad \boxed{D = \frac{\ln N}{\ln(1/\varepsilon)}}$$

We shall apply this formula to calculate the fractal dimension of the famous "*Cantor ternary set*", that is the most ancient known fractal. It was introduced by the german mathematician Georg Cantor (1845-1918), who is considered one of the

founders of set theory. To construct this set, let us begin with a given segment that is divided into three equal parts (Fig. 10.2) and leaving aside the middle third. Then the left and right thirds are again divided in three equal parts and the middle third is left aside. The process is repeated until after many iterations, we get discrete points that form the so called "*Cantor powder*". If we take the initial length equal to unity, after three iterations, we shall have  $2^3 = 8$  segments, each of them of length  $3^{-3} = 1/27$ . After  $n$  iterations there will be  $2^n$  segments, each of length  $3^{-n}$ . The total length of the restant segments is equal to  $(2/3)^n$ , a quantity that tends evidently to zero when  $n$  tends to infinity. This implies that the fractal dimension of the Cantor ternary set is

$$D = \frac{\ln N}{\ln(1/\varepsilon)} = \frac{\ln 2^n}{\ln(1/3^{-n})} = 0,6309...$$

This value is an irrational number, being nearer from one than from zero, and this is, in a certain sense, a measure of its irregularity.

M. S. El Naschie has carefully analyzed the relations existent among the Hausdorff dimension of Cantor sets of higher order and the Golden Mean and the Silver Mean [37], [38]. In particular, in Reference [39], he proved five important theorems, three of them main theorems (Bijection Theorem, Theorem of the Golden Mean and Generalized Fibonacci Theorem) and two auxiliary theorems (Silver Mean Theorem and Arithmetic Mean Theorem). These theorems are related to the notion of KAM instability<sup>2</sup> and the global chaos in hamiltonian ( that conserve the energy) physical systems.

Indeed, certain members of the MMF play a very important rol in relation to the stability of some orbits in the  $n$ -dimensional phase space. For example, it is a very well known fact that orbits with a "*winding number*" equal to the Golden Mean are the most stable -- the winding number measures the mean displacement of a certain angle at each iteration of a discrete dynamical system. Furthermore, the connection between the hyperbolic map and more general dynamical systems, is closely related to *period duplication* and the *Golden Mean route to chaos*. The empirical finding of period duplication in a certain physical phenomenon, as well as the existence of certain irrational ratios that produce the onset to chaos when this ratio is equal to the Golden Mean, are very well known in modern References (see References [3] and [21]).

The forbidden symmetries we have already encountered in analyzing quasi-crystals, like the symmetries of order eight and twelve, may also be generated by Cantor multiplicative sets of higher order, together with the Golden Mean [40].

---

<sup>2</sup>Kolmogorov (1954), Arnold (1963) and Moser (1967), proved what is today known as KAM theorem. This theorem states that the motion in the phase space of Classical Mechanics is neither completely regular nor completely irregular. but that the sort path depends sensibly from the initial conditions..

Comparing the terms of the secondary Fibonacci sequence (3.1), with the ternary Fibonacci sequence, defined by the relation

$$(10.2) \quad B_{n+1} = B_{n-2} + B_{n-1} + B_n ,$$

like it is indicated in the following table:

$n$	1	2	3	4	5	6	7	8	9
$F_n$	1	1	2	3	5	8	13	21	34
$B_n$	1	1	2	4	7	13	24	44	81

it is easy to verify that for the first sequence,  $F_n$  and  $n$  are equal only when  $n = 5$ , while for the second one,  $B_n$  and  $n$  are equal only when  $n = 4$ . These type of states is normally used to modellize some forms of *ergodic*<sup>3</sup> behavior of physical systems and they can be considered as “*ergodic-type states*”. The connections of this research with statistical mechanics, classic as well as quantum mechanics, as is proved by El Naschie [41], determine the existence of two types of quasi-ergodic Cantor sets:

- a) an even set of four dimensions, that describes the behavior of classical particles and *bosons*<sup>4</sup>;
- b) an odd set of five dimensions, related with *fermions*<sup>5</sup> and with the pentagonal symmetry of quasi-crystals.

## 11. TIME IRREVERSIBILITY

Ilya Prigogine is, without any doubt, one of the most important scientists of this century. He awarded the Nobel Prize in Chemistry and nowadays, he is the leader of a brilliant research group at the Free University in Brussel, Belgium. The fundamental question of time irreversibility and its consequences in science philosophy, has been one of his main preoccupations.

The basic laws in Physics, from newtonian Mechanics to the generalized relativity theory of Einstein, as well as the present theories for the elementary particles, satisfy all the hypothesis of time reversibility.

<sup>3</sup> In Dynamics, it is a very important problem to be able to describe the path of a particle in space. If the particle is limited to move inside a limited domain of space, it is essential to know if the path fills out all the space with an uniform distribution in a sufficiently long time. Such paths are called “*ergodic*” and to postulate their existence is a fundamental problem in classic Dynamics as well as in Quantum Mechanics.

<sup>4</sup> Bosons are elementary particles with a “*spin*” or angular momentum that is an integer multiple of Planck’s constant. Photons and mesons are bosons.

<sup>5</sup> Fermions are elementary particles with a “*spin*” that is a half-integer multiple of Planck’s constant. Electrons, protons and neutrons are fermions.

As Einstein stated: "*the distinction among past, present and future, is only an illusion*". However, time seems to flow in one sense. How is it possible to reconcile the **fundamental statement** with the **empirical fact**?

In his recently appeared book [42], Prigogine considers this question and the finding of an answer obliges him to revise and restate all the Physics, starting from Epicur's dilemma for whom the problem of the intelligibility of nature is undetachable from men destiny.

Together with Prigogine and other scientists, El Naschie proposes a solution valid for classical Mechanics as well as for Quantum Mechanics [43]. The solution consists in the introduction of the notion of a "*cantorian*" (from Cantor) space-time, in which time behaves statistically and is completely undistinguishable from the restant three space coordinates. What is really remarkable of this Cantorian space-time is that applying all the probabilistic necessary laws, the values of Hausdorff dimensions are intrinsically linked to the Golden Mean  $\phi$  and its successive powers, like  $\phi^2 = [2, \overline{1}]$  and  $\phi^3 = [\overline{4}]$  (see Reference [44])!

Obviously, Hausdorff dimension, being an intermediate measure between volume and dimension, plays in this new theory a preponderant rol as a linkage between dimension and information. We may as well conjecture a relation between the irrationality grade and the information content, since when the dimension is equal to the Golden Mean  $\phi$  — the most irrational of all irrational numbers -- the information content is the largest possible.

## 12. CONCLUSIONS

We have already verified how the MMF is closely related to the transition from a periodic dynamics to a quasi-periodic dynamics, as well as to the onset from order to chaos and with time irreversibility.

But simultaneously, since the beginning of humanity, there have been philosophical, natural and aesthetic considerations that have had primacy in the establishment of proportions based on some members of this family. They appeared more or less explicitly in the sacred art of Egypt, India, China and Islam and other ancient civilizations. They have dominated greek art and architecture, they persisted concealed in the monuments of the Gothic Middle Ages and re-emerged openly to be celebrated in the Renaissance.

Summarizing, we can state that wherever there is an intensification of function or a particular beauty and harmony of form, there at least the two first members of the MMF, e.g. the Golden Mean and the Silver Mean, will be found. If the restant members of this family are also involucred in these considerations, future researchs will give the answer.

Such a wide range of applications where the members of the MMF are present, opens the road to new inter-disciplinary investigations that undoubtedly will clear up the existent relations between Art and Technique, building a bridge linking the rational scientific approach and the aesthetic emotion. And perhaps this new perspective could help us in giving Technology, from which we depend each time more and more for our survival, a more human character.

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## FIGURE CAPTIONS

Fig. 5.1: Golden relations in a pentagon or unity side.

Fig. 5.2: Golden divisions in an ancient mask of Hermes.

Fig. 5.3: The Silver Mean in a regular octagon.

Fig. 6.1: Fractalization of a cube.

Fig. 6.2: Icosahedron.

Fig. 6.3: Icosahedral spiral.

Fig. 8.1a: Hyperbolic map starting from  $\pi$ .

Fig. 8.1b: Ordered sequence of 200 points for the hyperbolic map of Fig. 8.1a.

Fig. 8.2a: Hyperbolic map starting from  $e$ .

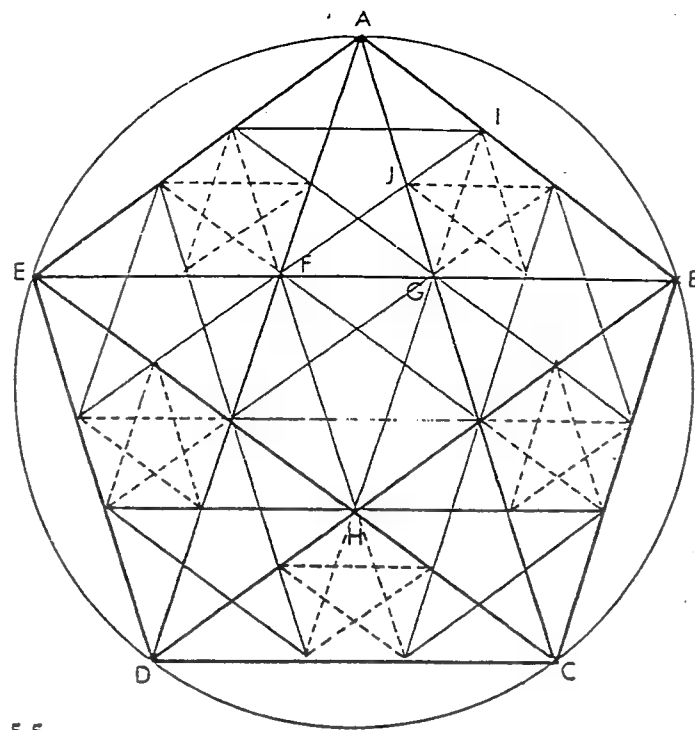
Fig. 8.2b: Ordered sequence of 200 points for the hyperbolic map of Fig. 8.2a.

Fig. 8.3: Hyperbolic map starting from the Golden Mean.

Fig. 8.4: Hyperbolic map starting from the other Metallic Means.

Fig. 9.1: Regular tilings for tessellating the plane.

Fig. 9.2: Cantor "powder" set.



5.5

Fig. 5.1

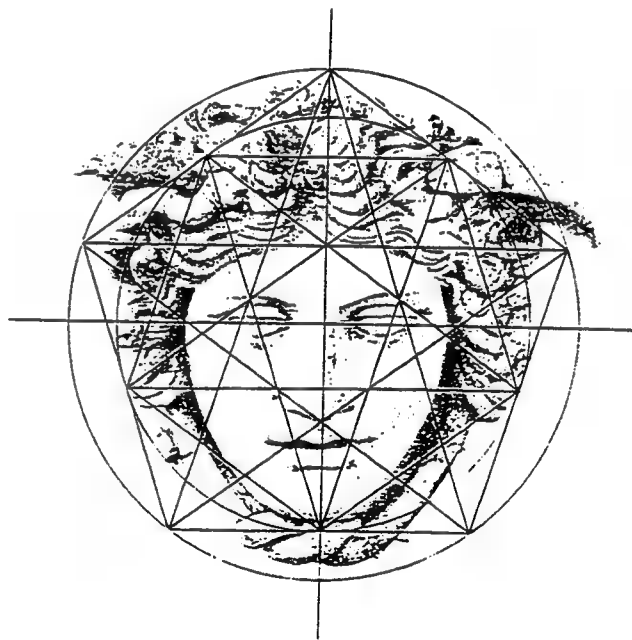


Fig. 5.2

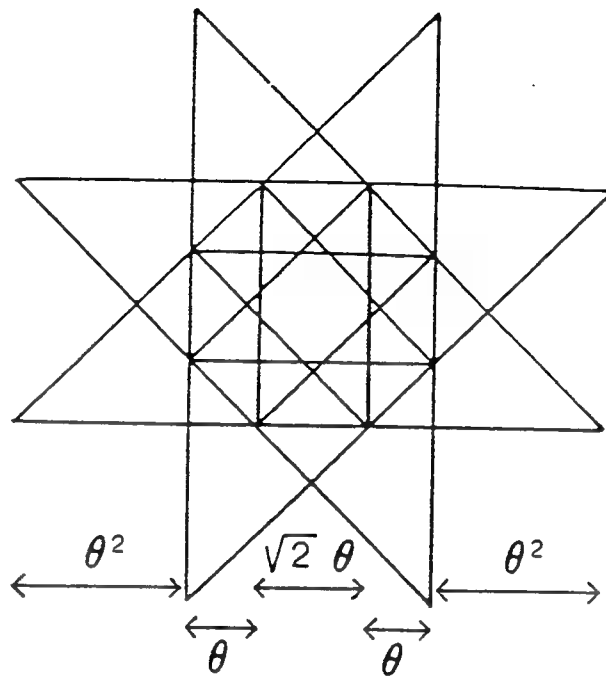


Fig. 5.3

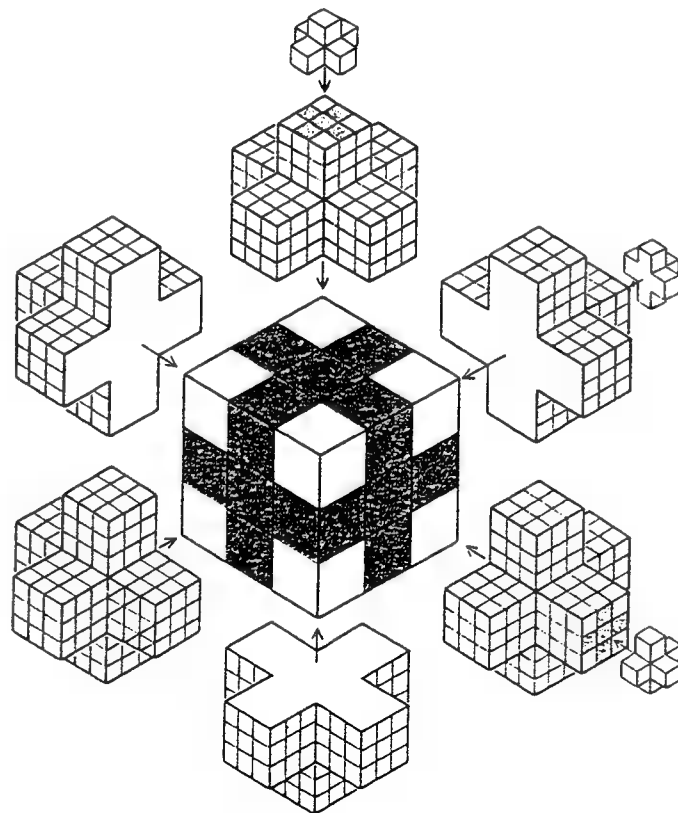


Fig. 6.1

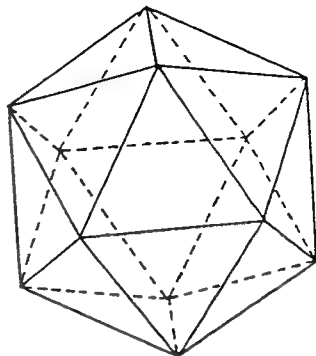


Fig. 6.2

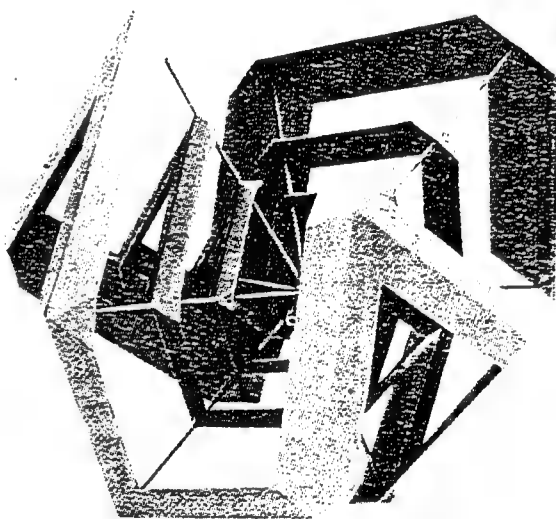


Fig. 6.3

```

> x:=evalf(P1);
> R:=[seq(H(),i=1..1000)];
> plot(R,0..1,0..1,style=POINT);

```

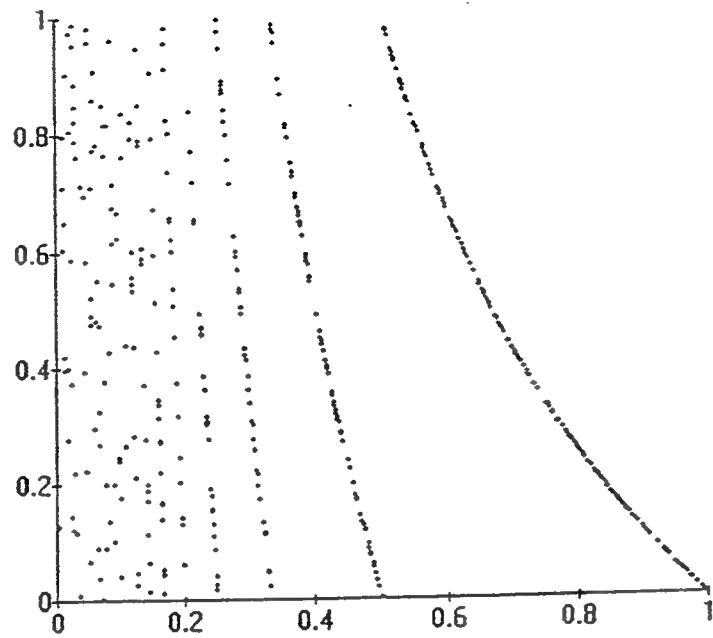


Fig. 8.1a

```

>
> R2:=[seq(H(),i=1..200)];
> plot(R2,0..1,0..1,style=LINE);

```

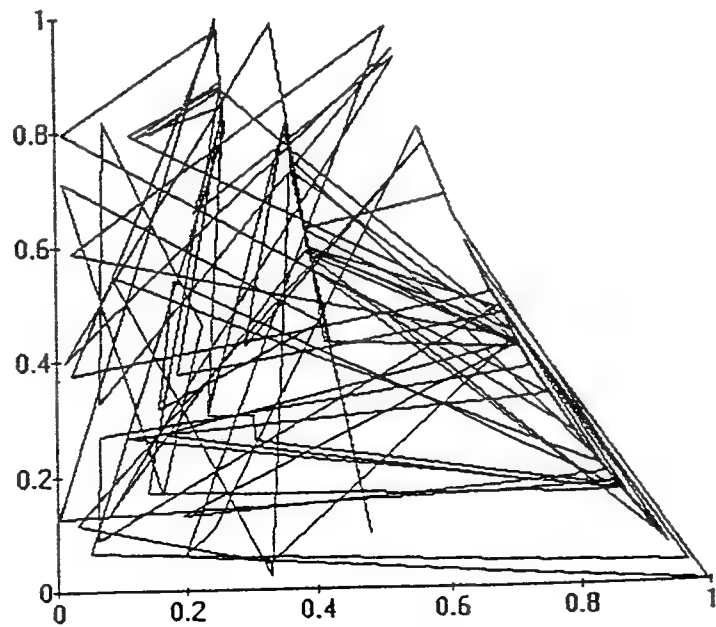


Fig. 8.1b

```

> x:=evalf(E);
      x:=2.7182818284590452353602874713526624977572470937
> R:=[seq(H(),i=1..1000)]:
> plot(R,0..1,0..1,style=POINT);

```

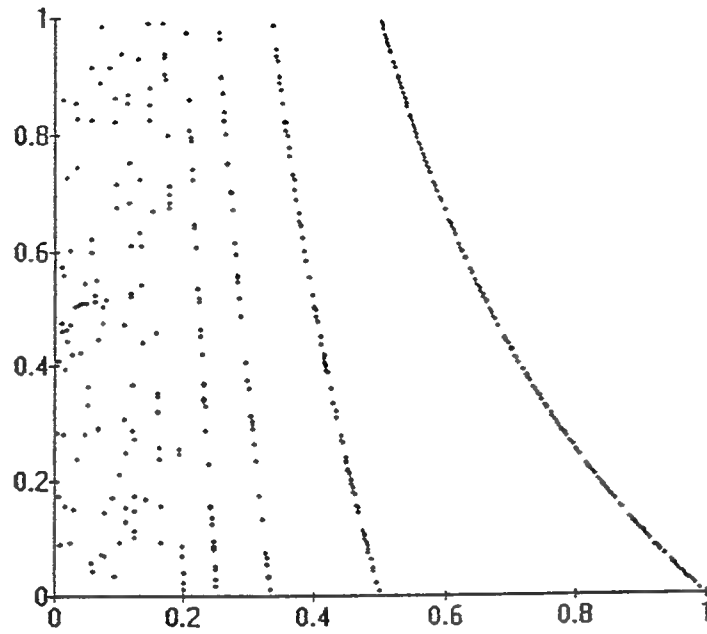


Fig. 8.2a

```

> R1:=[seq(H(),i=1..200)]:
> plot(R1,0..1,0..1,style=LINE);

```

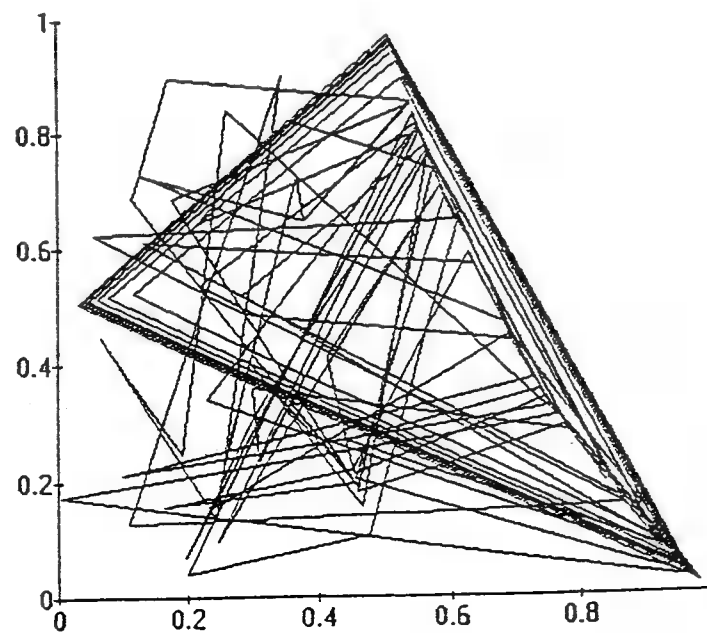


Fig. 8.2b

```

>
> Digits:=50;
> x:=evalf((1+sqrt(5))/2);
> F1:=array(1..1000);
> H1:=proc();
> if i=1 then F1[i]:=frac(x) else F1[i]:=frac(1/frac(F1[i-1]))
> fi;
> end;
> R1:=[seq(H1(),i=1..50)];
> R3:=[R1[2..49]];
> with(plots):
> RP:=plot(R1,0..1,0..1,style=POINT):
> RP3:=plot(R3,0..1,0..1,style=POINT):
> display({RP,RP3});

```

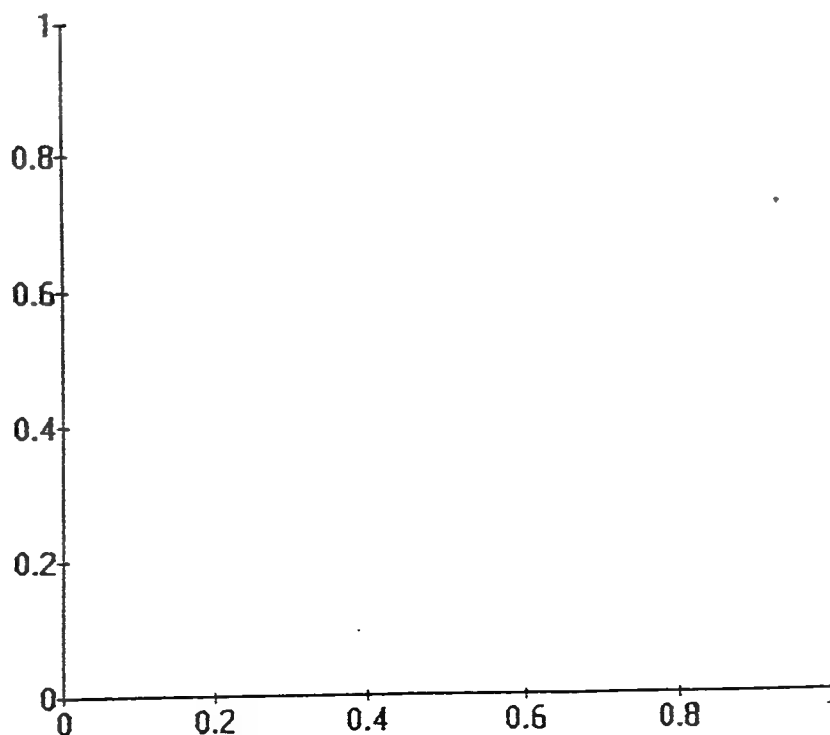


Fig. 8.3

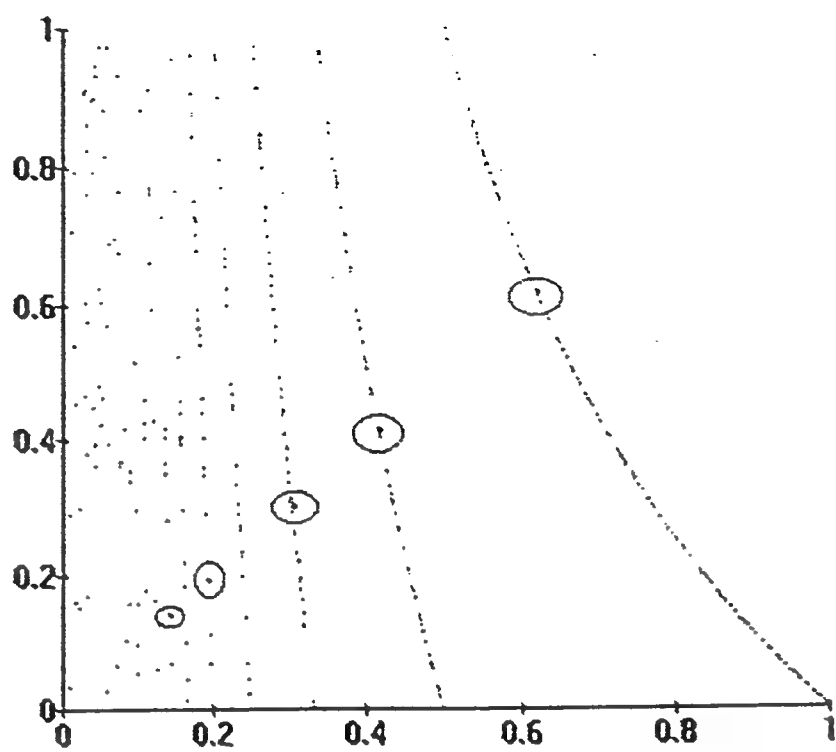


Fig. 8.4



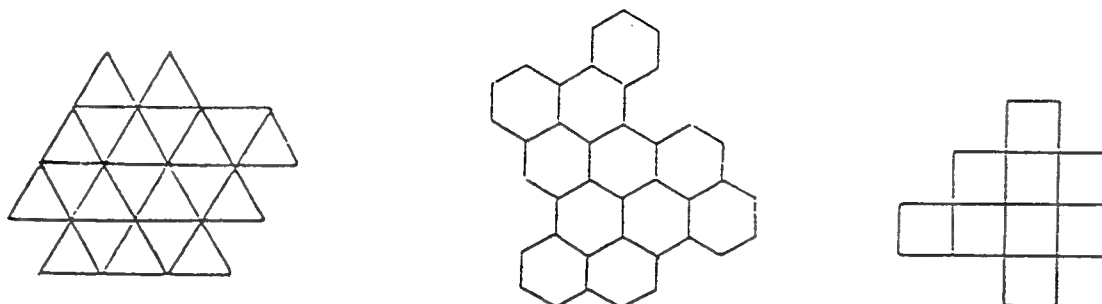


Fig. 9.1

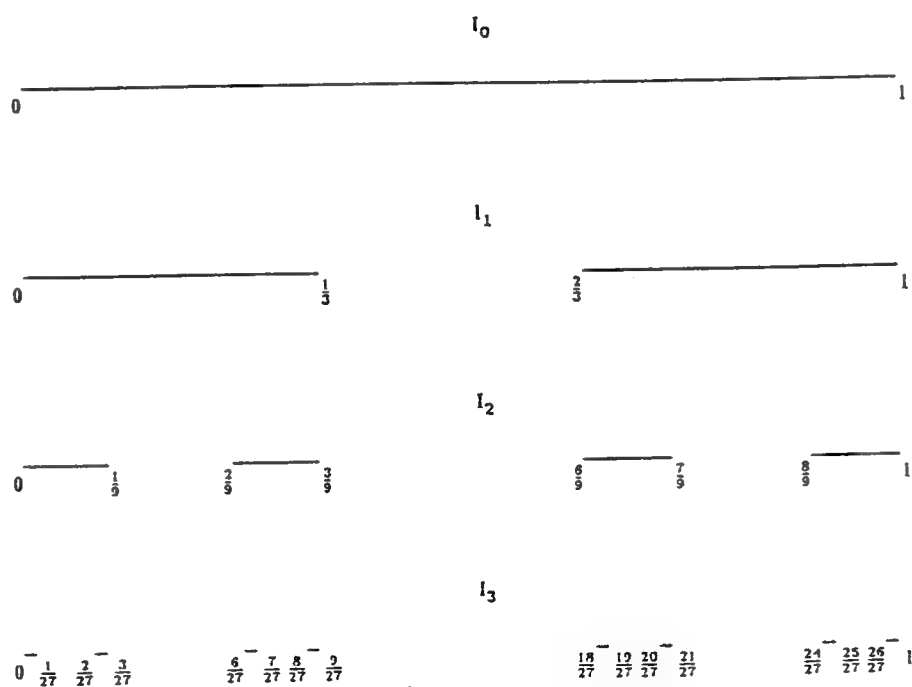


Fig. 9.2

# The equations $m \cdot S(m) = n \cdot S(n)$ and $m \cdot S(n) = n \cdot S(m)$ have infinitily many solutions

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Let be  $S : \mathcal{N}^* \rightarrow \mathcal{N}^*$  the Smarandache function.

$$S(n) = \min \{ k \mid n \leq_d k! \}$$

where  $\leq_d$  is the order generated by:

$$\wedge_d = \text{g.c.d.}$$

$$\vee_d = \text{s.c.m.}$$

on set  $\mathcal{N}^*$ .

It is known that  $\mathcal{M}_d = (\mathcal{N}^*, \wedge_d, \vee_d)$  is a lattice where 1 is the smallest element and 0 is the biggest element. The order  $\leq_d$  is defined like in any lattice by:

$$n \leq_d m \Leftrightarrow n \wedge_d m = n \Leftrightarrow n \vee_d m = m$$

or, in other terms:

$$n \leq_d m \Leftrightarrow n \mid m.$$

Next we will study two diophantine equations which contain the Smarandache function.

Reminding of two of the features of Smarandache's function which we will need further:

1. Smarandache's function satisfies:

$$S(m \vee_d n) = \max\{S(m), S(n)\}$$

2. To calculate  $S(p^\alpha)$ :

- 2.a. we will write the exponent in the generalized base  $[p]$  definite by the sequence with general term:

$$a_i(p) = \frac{p^i - 1}{p - 1}$$

who satisfies:

$$a_{i+1}(p) = p \cdot a_i(p) + 1$$

that is:

$$[p]: a_1(p), a_2(p), \dots$$

2.b. the result is read in the standard base  $(p)$  definite by the sequence:

$$b_i(p) = p^i$$

who satisfies:

$$b_{i+1}(p) = p \cdot b_i(p)$$

that is:

$$(p): 1, p, p^2, p^3, \dots$$

2.c. the number obtained will be multiply by  $p$ .

**Proposition:**

The equation

$$mS(m) = nS(n) \tag{1}$$

has infinity many solutions in the next two cases:

1.  $m = n$  - obvious
2.  $m > n$  with  $m = d \cdot a$ ,  $n = d \cdot b$  satisfying  $m \wedge_d n = d$ ,  $d \wedge_d a = 1$ ,  
 $d \wedge_d b > 1$  and the dual of this condition for  $m < n$ .

The equation

$$mS(n) = nS(m) \tag{2}$$

has infinity many solutions in the next two cases:

1.  $m = n$  - obvious
2.  $m > n$  and  $m \wedge_d n = 1$

**Proof**

Let's consider  $m > n$ . We distinguish the next cases:

1.  $m \wedge_d n = 1$  that is  $(m, n) = 1$ .

Then from equation (1) we can deduce:  $m \leq_d S(n)$ ; then  $m \leq S(n)$ . But  $S(n) \leq n$  for every  $n$  and as  $n < m$  we get the contradiction:  $S(n) < m$ .

For the equation (2) we have:  $m \leq_d S(m) \Rightarrow m \leq S(m) \Rightarrow m = S(m) \Rightarrow m = 4$  or  $m$  is ■ prime number. If  $m = 4$  the equation becomes:

$$4 \cdot S(n) = 4 \cdot n \Rightarrow n = S(n) \Rightarrow n = 4 \text{ or } n \text{ is a prime number}$$

So in this case the equation has for solutions the pairs of numbers:

$$(4,4), (4,p), (p,4), (p,q) \text{ with } p,q \text{ prime numbers.}$$

2. If  $m \wedge_d n = d \neq 1$ , so:

$$\begin{cases} m = d \cdot a \\ n = d \cdot b \end{cases}, \text{ cu } a \wedge_d b = 1 \quad (3)$$

the equation (1) becomes:

$$a \cdot S(m) = b \cdot S(n) \quad (4)$$

From condition  $m > n$  we deduce:

$$a > b$$

We can distinguish the next possibilities:

$$\text{a) } d \wedge_d a = 1, d \wedge_d b = 1$$

If we note:

$$\mu = S(m), \nu = S(n)$$

we have:

$$\begin{aligned} \mu &= S(m) = S(d \cdot a) = S(d \vee_d a) = \max(S(d), S(a)) \\ \nu &= S(n) = S(d \cdot b) = S(d \vee_d b) = \max(S(d), S(b)) \end{aligned} \quad (5)$$

and the equation (1) is equivalent with:

$$\frac{m}{n} = \frac{S(n)}{S(m)} \Leftrightarrow \frac{a}{b} = \frac{\nu}{\mu} \quad (6)$$

From (5) we deduce for  $\mu$  and  $\nu$  the possibilities:

$$\text{a1) } \mu = S(d), \nu = S(d), \text{ that is:}$$

$$S(d) \geq S(a) \text{ and } S(d) \geq S(b)$$

In this case (6) becomes:

$$\frac{a}{b} = 1 - \text{false}$$

$$\text{a2) } \mu = S(d), \nu = S(b), \text{ that is:}$$

$$S(d) \geq S(a) \text{ and } S(d) < S(b)$$

In this case (6) becomes:

$$\frac{a}{b} = \frac{S(b)}{S(d)} \Rightarrow aS(d) = bS(b)$$

But  $a \wedge_d b = 1$ , so we must have:

$$a \leq_d S(b) \text{ so } a \leq S(b) \quad (7)$$

and in the same time:

$$S(b) \leq b < a - \text{contradicts (7)}$$

**a3)**  $\mu = S(a)$ ,  $\nu = S(d)$  that is:

$$S(a) > S(d) \text{ and } S(d) \geq S(b) \quad (8)$$

In this case the equation (6) is:

$$\frac{a}{b} = \frac{S(d)}{S(a)}$$

that is:

$$aS(a) = bS(d) \quad (9)$$

Then from  $a \wedge_d b = 1 \Rightarrow a \leq_d S(d)$  and  $b \leq_d S(a)$ . So:

$$S(a) \leq a \leq S(d) - \text{contradicts (8)}$$

**a4)**  $\mu = S(a)$ ,  $\nu = S(b)$

In this case the equation (6) becomes:

$$\frac{a}{b} = \frac{S(b)}{S(a)} \text{ with } a \wedge_d b = 1$$

and we are in the case 1.

For the equation (2) which can be also write:

$$aS(n) = bS(m) \quad (10)$$

that is:  $a\nu = b\mu$

♦ in the conditions **a1)** it becomes:

$$a = b - \text{false}$$

♦ in the conditions **a2)** it becomes:

$$aS(b) = bS(d)$$

and as  $a \wedge_d b = 1$  we deduce:

$$a \leq_d S(d), \quad b \leq_d S(b).$$

So  $b \leq S(b)$ , that is  $b = S(b)$ , so  $b = 4$  or  $b = p$  - prime number

and the equation becomes:

$$a = S(d)$$

and as  $S(d) \wedge_d d > 1$  we obtain the contradiction:

$$a \wedge_d d > 1$$

♦ in the conditions **a3)** it becomes:

$$aS(d) = bS(a)$$

and because  $a \wedge_d b = 1$  we must have  $a \leq_d S(a)$  that is  $a = S(a)$ .

So the equation is:

$$S(d) = b$$

As  $d \wedge_d S(d) > 1$  it results  $d \wedge_d b > 1$  - false.

♦ in the conditions a4) the equation becomes:

$$aS(b) = bS(a)$$

that is the equation (2) in the case 1.

**b)**  $d \wedge_d a > 1$  and  $d \wedge_d b = 1$

As (1) is equivalent with (4) from  $a \wedge_d b = 1$  it results:

$$a \leq_d S(n) \text{ and } b \leq_d S(m)$$

From the hypothesis ( $d \wedge_d a > 1$ ) it results:

$$S(m) = S(a \cdot d) \geq \max\{S(a), S(d)\} \quad (11)$$

If in (11) the inequality is not top, that is:

$$S(m) = \max\{S(d), S(a)\}$$

and as

$$S(n) = \max\{S(d), S(b)\} \quad (12)$$

we are in the in the case a). Let's suppose that in (11) the inequality is top:

$$S(m) > \max\{S(a), S(d)\}$$

It results:

$$S(m) > S(a) \quad (13)$$

$$S(m) > S(d) \quad (14)$$

Reminding of (11) we have the next cases:

**b1)**  $S(n) = S(d)$

The equation(4) becomes:

$$aS(m) = bS(d)$$

and from  $a > b$  it results  $S(d) > S(m)$  - false (13).

**b2)**  $S(n) = S(b)$

The equation (4) becomes:

$$aS(m) = bS(b)$$

As  $\gcd(a, b) = 1$  it results  $a \leq_d S(b)$  so  $a \leq S(b)$  - false because

$$S(b) \leq b < a.$$

**c)**  $d \wedge_d a = 1$  and  $d \wedge_d b > 1$

We get:

$$S(m) = S(d \cdot a) = S(d \vee^d a) = \max\{S(d), S(a)\}$$

$$S(n) = S(d \cdot b) \geq \max\{S(d), S(b)\}$$

If the last inequality is not top, we have the case a). So let it be:

$$S(n) > \max\{S(d), S(b)\},$$

that is:

$$S(n) > S(d) \quad (15)$$

and

$$S(n) > S(b) \quad (16)$$

**c1)**  $S(m) = S(d)$ , that is  $S(d) \geq S(a)$ . The equation becomes:

$$aS(d) = bS(n)$$

We can't get a contradiction and we can see that the equation has solutions like this:

$$m = p^a \cdot a$$

$$n = p^{a+x}$$

So  $b = p^x$ ,  $d = p^a$ . The condition  $a > b$  becomes  $a > p^x$ . We must have also

$$a \wedge_d p^a = 1, \text{ that is } a \wedge_d p = 1.$$

The equation becomes:

$$p^a a \cdot S(p^a) = p^{a+x} S \cdot (p^{a+x})$$

It results:

$$a = \frac{p^x S(p^{a+x})}{S(p^a)} = \frac{p^x p((\alpha+x)_{[p]}_{(p)})}{p(\alpha_{[p]}_{(p)})} = \frac{p^x ((\alpha+x)_{[p]}_{(p)})}{(\alpha_{[p]}_{(p)})}.$$

We can see that choosing  $\alpha$  this way:

$$(\alpha_{[p]}_{(p)}) = p^x = (\underbrace{100\dots 0}_{x \text{ times}})_{(p)} \Rightarrow \alpha = \alpha_{[p]} = (\underbrace{100\dots 0}_{x \text{ times}})_{[p]} = a_{x+1}(p)$$

we get:

$$a = ((\alpha+x)_{[p]}_{(p)}) \in N$$

We must also put the condition  $a \wedge_d p = 1$  which we can get choosing convenient values for  $x$ .

Example: For  $n = 3$  we have:

$$(3): 1, 3, 9, 27, \dots$$

$$[3]: 1, 4, 13, 40, 121, \dots$$

Considering  $x = 2$  we get (from condition  $(\alpha_{[p]}_{(p)}) = p^x$ ):

$$(\alpha_{[3]})_{(3)} = 3^x = 3^2 = 100_{(3)} \Rightarrow \alpha = 100_{[3]} = 13 \Rightarrow$$

$$a = S(p^{\alpha+x}) = S(3^{13+2}) = S(3^{15}) = (15_{[3]})_{(3)} = 102_{(3)} = 11$$

So,  $(m = 3^{13} \cdot 11, n = 3^{15})$  is solution for equation (1).

Equation (2) which has the form:

$$aS(n) = bS(d)$$

has no solutions because from  $a > b \Rightarrow S(d) > S(n)$  - false.

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# On The Irrationality Of Certain Alternative Smarandache Series

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1. Let  $S(n)$  be the Smarandache function. In paper [1] it is proved the irrationality of  $\sum_{n=1}^{\infty} \frac{S(n)}{n!}$ . We note here that this result is contained in the following more general theorem (see e.g. [2]).

**Theorem 1** Let  $(x_n)$  be a sequence of natural numbers with the properties: (1) there exists  $n_0 \in \mathbb{N}^*$  such that  $x_n \leq n$  for all  $n \geq n_0$ ; (2)  $x_n < n-1$  for an infinity of  $n$ ; (3)  $x_m > 0$  for infinitely many  $m$ . Then the series  $\sum_{n=1}^{\infty} \frac{x_n}{n!}$  is irrational.

By letting  $x_n = S(n)$ , it is well known that  $S(n) \leq n$  for  $n \geq n_0 \equiv 1$ , and  $S(n) \leq \frac{2}{3}n$  for  $n > 4$ , composite. Clearly,  $\frac{2}{3}n < n-1$  for  $n > 3$ . Thus the irrationality of the second constant of Smarandache ([1]) is contained in the above result.

2. We now prove a result on the irrationality of the alternating series  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{S(n)}{n!}$ .

We can formulate our result more generally, as follows:

**Theorem 2** Let  $(a_n), (b_n)$  be two sequences of positive integers having the following properties: (1)  $n | a_1 a_2 \dots a_n$  for all  $n \geq n_0$  ( $n_0 \in \mathbb{N}^*$ ); (2)  $\frac{b_{n+1}}{a_{n+1}} < b_n \leq a_n$  for  $n \geq n_0$ ; (3)  $b_m < a_m$ , where  $m \geq n_0$  is composite. Then the series  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{b_n}{a_1 a_2 \dots a_n}$  is convergent and has an irrational value.

**Proof:** It is sufficient to consider the series  $\sum_{n=n_0}^{\infty} (-1)^{n-1} \frac{b_n}{a_1 a_2 \dots a_n}$ . The proof is very similar (in some aspect) to Theorem 2 in our paper [3]. Let  $x_n = \frac{b_n}{a_1 a_2 \dots a_n}$  ( $n \geq n_0$ ).

Then  $x_n \leq \frac{1}{a_1 \dots a_{n-1}} \rightarrow 0$  since (1) gives  $a_1 \dots a_k \geq k \rightarrow \infty$  (as  $k \rightarrow \infty$ ). On the other hand,  $x_{n+1} < x_n$  by the first part of (2). Thus Leibnitz criteria assures the convergence of the series. Let us now assume, on the contrary, that the series has a rational value, say  $\frac{a}{k}$ . First we note that we can choose  $k$  in such a manner that  $k+1$  is composite, and  $k > n_0$ . Indeed, if  $k+1 = p$  (prime), then  $\frac{a}{p-1} = \frac{ca}{c(p-1)}$ . Let  $c = 2ar^2 + 2r$ , where  $r$  is arbitrary. Then  $2a(2ar^2 + 2r) + 1 = (2ar + 1)^2$ , which is composite. Since  $r$  is arbitrary, we can assume  $k > n_0$ . By multiplying the sum with  $a_1 a_2 \dots a_k$ , we can write:

$$a \frac{a_1 \dots a_k}{k} = \sum_{n=n_0}^k (-1)^{n-1} \frac{a_1 \dots a_k}{a_1 \dots a_n} \cdot b_n + (-1)^k \left( \frac{b_{k+1}}{a_{k+1}} - \frac{b_{k+2}}{a_{k+1} a_{k+2}} + \dots \right).$$

The alternating series on the right side is convergent and must have an integer value. But it is well known its value lies between  $\frac{b_{k+1}}{a_{k+1}} - \frac{b_{k+2}}{a_{k+1} a_{k+2}}$  and  $\frac{b_{k+1}}{a_{k+1}}$ . Here  $\frac{b_{k+1}}{a_{k+1}} - \frac{b_{k+2}}{a_{k+1} a_{k+2}} > 0$  on base of (3). On the other hand  $\frac{b_{k+1}}{a_{k+1}} < 1$ , since  $k+1$  is a composite number. Since an integer number has a value between 0 and 1, we have obtained a contradiction, finishing the proof of the theorem.

**Corollary**  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{S(n)}{n!}$  is irrational.

**Proof:** Let  $a_n = n$ . Then condition (1) of Theorem 2 is obvious for all  $n$ ; (2) is valid with  $n_0 = 2$ , since  $S(n) \leq n$  and  $S(n+1) \leq n+1 = (n+1) \cdot 1 < (n+1)S(n)$  for  $n \geq 2$ .

For composite  $m$  we have  $S(m) \leq \frac{2}{3}m < m$ , thus condition (3) is verified, too.

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# Some Elementary Algebraic Considerations Inspired by Smarandache Type Functions

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The basic idea of this paper is the algebraic construction of some functions representing prolongations of the Smarandache type functions to more complete sets already known and having specified properties.

A. Starting from a sequence of positive integers  $\sigma : \mathbb{N}^* \rightarrow \mathbb{N}^*$  satisfying the condition

$$\forall n \in \mathbb{N}^*, \exists m_n \in \mathbb{N}^*, \forall m \geq m_n \implies n/\sigma(m) \quad (1)$$

(such sequences-possibly satisfying an extra condition-considered by C. Christol to generalise the  $p$ -adic numbers were called also multiplicative convergent to zero; for example:  $\sigma(n) = n!$ ) it was built an associated Smarandache type function that is  $S_\sigma : \mathbb{N}^* \rightarrow \mathbb{N}^*$  defined by

$$S_\sigma(n) = \min \{m_n : m_n \text{ is given by (1)}\} \quad (2)$$

(For  $\sigma : \mathbb{N}^* \rightarrow \mathbb{N}^*$  with  $\sigma(n) = n!$  the associated function  $S_\sigma$  is just the Smarandache function.)

The sequence is noted  $\sigma_{0d}$  and the associated function  $S_{0d}$ .

For each such a sequence, the associated function has a series of properties already proved, from whom we retain:

$$S_{0d}([n_1, n_2]) = \max \{S_{0d}(n_1), S_{0d}(n_2)\} \quad (\text{see [1, th. 2.2]}). \quad (3)$$

We can stand out:-the universal algebra  $(\mathbb{N}^*, \Omega)$ , the set of operations is  $\Omega = \{\vee_d, \varphi_0\}$  where  $\vee_d : (\mathbb{N}^*)^2 \rightarrow \mathbb{N}^*$  with  $\forall x, y \in \mathbb{N}^*, x \vee_d y = [x, y]$  and

$\varphi_0 : (N^*)^2 \rightarrow N^*$  the null operation that fixes 1-unique particular element with the role of neutral element for " $\vee_d$ ";  $1 = e_{\vee_d}$  -the universal algebra  $(N^*, \Omega')$  with  $\Omega' = \{\vee, \psi_0\}$  where  $\vee : (N^*)^2 \rightarrow N^*$ ,  $\forall x, y \in N^*$ ;  $x \vee y = \sup \{x, y\}$  and  $\psi_0 : (N^*)^2 \rightarrow N^*$  a null operation that fixes 1-unique particular element with the role of neutral element for " $\vee$ ":  $1 = e_{\vee}$ . We observe that the universal algebra  $(N^*, \Omega)$  and  $(N^*, \Omega')$  are of the same type

$$\begin{pmatrix} \vee_d & \varphi_0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} \vee & \psi_0 \\ 2 & 0 \end{pmatrix}$$

and with the similarity (bijective)  $\vee_d \iff \vee$  and  $\varphi_0 \iff \psi_0$  function  $S_{0d} : N^* \rightarrow N^*$  is a morphism between them.

We already know that  $((N^*)^I, \overline{\Omega})$  with  $I$ -a some set-is an universal algebra with  $\overline{\Omega} = \{w_1, w_0\}$  defined by :

$$w_1 : (N^*)^I \times (N^*)^I \rightarrow (N^*)^I$$

with

$$\forall a = \{a_i\}_{i \in I}, b = \{b_i\}_{i \in I}, a, b \in (N^*)^I, w_1(a, b) = \{a_i \vee_d b_i\}_{i \in I}$$

and  $w_0$  a null operation:  $e_{w_1} = \{e_i = 1\}_{i \in I}$  (the canonical projections  $\tilde{p}_j$  being, of course, morphismes between  $((N^*)^I, \overline{\Omega})$  and  $(N^*, \Omega)$  (see [3, th. 1.a)]).

We also know that  $((N^*)^I, \Omega')$  with  $\Omega' = \{\theta_1, \theta_0\}$  defined by

$$\theta_1 : (N^*)^I \times (N^*)^I \rightarrow (N^*)^I$$

by

$$\forall a = \{a_i\}_{i \in I}, b = \{b_i\}_{i \in I}, a, b \in (N^*)^I, \theta_1(a, b) = \{a_i \vee b_i\}_{i \in I}$$

and  $\theta_0$  - a null operation:  $e_{\theta_1} = \{e_i = 1\}_{i \in I}$  (neutral element) is an universal algebra and is of the same type as the above one.

With all these known elements we can state:

**Theorem 1** *If  $S_{0d} : N^* \rightarrow N^*$  is a Smarandache type function defined by (2), morphism between  $(N^*, \Omega)$  and  $(N^*, \Omega')$  and  $I$  is a some set, then there is an unique  $s_{0d} : (N^*)^I \rightarrow (N^*)^I$ , morphism between the universal algebras  $((N^*)^I, \overline{\Omega})$  and  $((N^*)^I, \overline{\Omega}')$  so that  $p_i \circ s_{0d} = S_{0d} \circ \tilde{p}_i$ ,  $i \in I$ , where  $p_j : (N^*)^I \rightarrow (N^*)$  with  $\forall a = \{a_i\}_{i \in I} \in (N^*)^I$ ,  $p_j(a) = a_j$ ,  $\forall j \in I$  are the canonical projections, morphismes between  $((N^*)^I, \overline{\Omega}')$  and  $(N^*, \Omega')$ ,  $\tilde{p}_i : (N^*)^I \rightarrow N^*$ , analogous between  $((N^*)^I, \overline{\Omega})$  and  $(N^*, \Omega)$ .*

**Proof.** We use the property of universality of the universal algebra  $((N^*)^I, \bar{\Omega})$ : for every  $(A, \bar{\theta})$  with  $\bar{\theta} = \{T, \bar{\theta}_0\}$  is an universal algebra of the same type with  $((N^*)^I, \bar{\Omega})$  and  $u_i : A \rightarrow N^*, \forall i \in I$ , morphismes between  $(A, \bar{\theta})$  and  $(N^*, \Omega')$ , there is an unique  $u : A \rightarrow (N^*)^I$  morphism between the universal algebras  $(A, \bar{\theta})$  and  $((N^*)^I, \bar{\Omega})$ , so that  $p_j \circ u = u_j, \forall j \in I$ , with  $p_j$  - the canonical projections. A some universal algebra can be  $((N^*)^I, \bar{\Omega})$  because is of the same type and the morphismes from the assumption can be  $u_i : (N^*)^I \rightarrow N^*$  defined by:

$$\forall a = \{a_i\}_{i \in I} \in (N^*)^I, u_j(a) = S_{0d}(a_j) \iff u_j = S_{0d} \circ \tilde{p}_j, \forall j \in I,$$

where  $S_{0d}$  is a Smarandache type function, morphisme, as we know from (3) and  $\tilde{p}_j$  - the canonical projections, morphismes between  $((N^*)^I, \bar{\Omega})$  and  $(N^*, \Omega)$  ( $u_i$  are morphismes as a composition of two morphismes). The assumptions of the property of universality being ensured, it results that there is an unique  $s_{0d} : (N^*)^I \rightarrow (N^*)^I$ , morphismes between  $((N^*)^I, \bar{\Omega})$  and  $((N^*)^I, \bar{\Omega})$  so that  $p_j \circ s_{0d} = u_j, \forall j \in I$ , i.e.  $p_j \circ s_{0d} = S_{0d} \circ \tilde{p}_j, \forall j \in I$ . ■

**B.** A sequence of positive integers  $\sigma : N^* \rightarrow N^*$  is called " of divisibility (d.s.)" if:

$$m/n \implies \sigma(m) / \sigma(n) \quad (4)$$

and "of strong divisibility (s.d.s.)" if:

$$\sigma((m, n)) = (\sigma(m), \sigma(n)), \forall m, n \in N^*, \quad (5)$$

with  $(m, n)$  the greatest common factor.

The sequence s.d.s. were studied by N. Jansen; the Fibonacci sequence defined by

$$F_{n+1} = F_n + F_{n-1} \text{ with } F_1 = F_2 = 1$$

is a s.d.s.

Starting from a sequence  $\sigma_{dd} : N^* \rightarrow N^*$  that satisfies the condition

$$\forall n \in N^*, \exists m_n \in N^*, \forall m \in N^*, m_n/n \implies n/\sigma(m), \quad (6)$$

as associated Smarandache's function was built that is  $S_{dd} : N^* \rightarrow N^*$  given by

$$S_{dd}(n) = \min \{m_n : m_n \text{ is given by (6)}\}, \forall n \in N^*, \quad (7)$$

having a series of already known properties from which we retain:  
if the sequence  $\sigma_{dd}$  is s.d.s. and satisfies (6), then

$$S_{dd}([n_1, n_2]) = [S_{dd}(n_1), S_{dd}(n_2)], \forall n_1, n_2 \in \mathbb{N}^*, \quad (8)$$

where  $[a, b]$  is the smallest common multiple of  $a$  and  $b$  (see [1, th. 2.5]).

We can stand out the universal algebra  $(\mathbb{N}^*, \Omega)$  where, this time,  $\Omega = \{\vee_d, \wedge_d, \varphi_0\}$  of the type  $\tau = \begin{pmatrix} \vee_d & \wedge_d & \varphi_0 \\ 2 & 2 & 0 \end{pmatrix}$  with known  $\vee_d$  and  $\varphi_0$  (from A) and  $\wedge_d : (\mathbb{N}^*)^2 \rightarrow \mathbb{N}^*$  defined by

$$x \wedge_d y = (x, y), \forall x, y \in \mathbb{N}^*.$$

It is known that then there is an universal algebra  $((\mathbb{N}^*)^I, \overline{\Omega})$  with  $I$  - a some set and here  $\overline{\Omega} = \{w_1, w_2, w_0\}$  with  $w_1, w_0$  known and  $w_2 : (\mathbb{N}^*)^I \times (\mathbb{N}^*)^I \rightarrow (\mathbb{N}^*)^I$  defined by:

$$w_2(a, b) = \{a_i \wedge_d b_i\}_{i \in I}, \forall a = \{a_i\}_{i \in I}, b = \{b_i\}_{i \in I} \in (\mathbb{N}^*)^I.$$

It can be stated the same as at A:

**Theorem 2** *If  $S_{dd} : \mathbb{N}^* \rightarrow \mathbb{N}^*$  is a Smarandache type function defined by (7), endomorphism for the universal algebra  $(\mathbb{N}^*, \Omega)$  and  $I$  - a some set, then there is an unique  $s_{dd} : (\mathbb{N}^*)^I \rightarrow (\mathbb{N}^*)^I$ , an endomorphism for the above universal algebra  $((\mathbb{N}^*)^I, \Omega)$  so that  $p_i \circ s_{dd} = S_{dd} \circ p_i, \forall i \in I$ .*

The analogical proof with that of th. 1. can be also done directly; the correspondence  $s_{dd}$  is defined and it is shown that is a function, endomorphism, the required conditions being obviously satisfied.

**Remark 1** *If the initial sequence  $\sigma_{dd}$  isn't at all s.d.s. but satisfies (6) with a view to the properties of the associated function, a function can be always defined  $s_{dd} : (\mathbb{N}^*)^I \rightarrow (\mathbb{N}^*)^I$  that is no more an endomorphism for the given universal algebra  $((\mathbb{N}^*)^I, \overline{\Omega})$  than in certain conditions or in particular cases (see [1, th. 2.4.]).*

C. Starting from a sequence noted  $\sigma_{d0}$  of positive integers  $\sigma_{d0} : \mathbb{N}^* \rightarrow \mathbb{N}^*$  that satisfies the condition:

$$\forall n \in \mathbb{N}^*, \exists m_n \in \mathbb{N}^*; \forall m \in \mathbb{N}^*, m_n/m \implies n \leq \sigma_{d0}(m) \quad (9)$$

are associated Smarandache type function was built, defined by:

$$S_{d0}(n) = \min \{m_n \setminus m_n \text{ satisfies (9)}\} \quad (10)$$

having known properties.

Standing out the universal algebra  $(\mathbb{N}^*, \Omega')$  when here  $\Omega' = \{\vee, \wedge, \dagger_0\}$  with  $\vee, \dagger_0$  known, and  $\wedge : (\mathbb{N}^*)^2 \rightarrow \mathbb{N}^*$  by

$$x \wedge y = \inf \{x, y\}, \forall x, y \in \mathbb{N}^*$$

it can be proved the same way that there is an unique  $s_{d0} : (\mathbb{N}^*)^I \rightarrow (\mathbb{N}^*)^I$  endomorphism of the universal algebra  $((\mathbb{N}^*)^I, \overline{\Omega'})$  so that

$$\tilde{p}_i \circ s_{d0} = S_{d0} \circ \tilde{p}_i, \forall i \in I.$$

Above we built the prolongations  $s_{ij}$  to more complexe sets of the Smarandache type functions noted  $S_{ij}$  (for  $I = \{1\} \implies s_{\underline{I}} = S_{ij}$ ). The algebraic properties of the  $s_{ij}$ , for their restrictions to  $\mathbb{N}^*$ , could bring new properties for the Smarandache type function that we considered above.

## References

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# CONSTRUCTION OF ELEMENTS OF THE SMARANDACHE SQUARE-PARTIAL-DIGITAL SUBSEQUENCE

by Lamarr Widmer

The Smarandache Square-Partial-Digital Subsequence (SPDS) is the sequence of square integers which admit a partition for which each segment is a square integer. An example is  $506^2 = 256036$  which has partition  $256|0|36$ . Ashbacher considers these numbers on page 44 of [1] and quickly shows that the SPDS is infinite by exhibiting two infinite "families" of elements. We will extend his results by showing how to construct infinite families of elements of SPDS containing desired patterns of digits.

Theorem 1: Let  $c$  be any concatenation of square numbers. There are infinitely many elements of SPDS which contain the sequence  $c$ .

proof: If  $c$  forms an even integer, let  $N = c$ . Otherwise, let  $N$  be  $c$  with a digit 4 added at the right. So  $N$  is an even number.

Find any factorization  $N = 2ab$ . Consider the number

$m = a \cdot 10^n + b$  for sufficiently large  $n$ . (Sufficiently large means that  $10^n > b^2$  and  $10^n > N$ .) Then  $m^2 = a^2 10^{2n} + N \cdot 10^n + b^2 \in \text{SPDS}$ .

Q.E.D.

For example, let us construct elements of SPDS containing the string  $c = 2514936$ . In the notation of our proof, we have  $ab = 1257468$  and we can use  $a = 6$  and  $b = 209578$  (among many possibilities). This gives us the numbers

$$600000209578^2 = 360000251493643922938084$$

$$6000000209578^2 = 36000002514936043922938084$$

$$60000000209578^2 = 3600000025149360043922938084$$

etc., which all belong to SPDS.

This allows us to imbed any sequence of squares in the interior of an element of SPDS. What about the ends? Clearly we cannot put all such sequences at the end of an element of SPDS. No perfect square ends in the digits 99, for example. Our best result in this respect is the following.

Theorem 2: Let  $a$  be any positive integer. There are infinitely many elements of SPDS which begin and end with  $a^2$ .

proof: For large enough  $n$  (ie.  $10^n > 225a^2$ ), consider

$$m = a \cdot 10^{2n} + \frac{a}{2} 10^n + a = a \cdot 10^{2n} + 5a \cdot 10^{n-1} + a$$

Then

$$\begin{aligned} m^2 &= a^2 \cdot 10^{4n} + a^2 \cdot 10^{3n} + \frac{9}{4} a^2 \cdot 10^{2n} + a^2 \cdot 10^n + a^2 \\ &= a^2 \cdot 10^{4n} + a^2 \cdot 10^{3n} + (15a)^2 \cdot 10^{2n-2} + a^2 \cdot 10^n + a^2 \end{aligned}$$

belongs to SPDS.

Q.E.D.



We illustrate for  $a = 8$ . For successive values of  $n$  beginning with 5, we have the following elements of SPDS.

$$80000400008^2 = 6400064001440006400064$$

$$8000004000008^2 = 64000064000144000064000064$$

$$800000040000008^2 = 640000064000014400000640000064$$

etc.

We have a number of observations concerning this last result. First, an obvious debt is owed to Ashbacher's work [1], in which he gives the family  $212^2 = 44944$ ,  $20102^2 = 404090404$ , .... Second, we actually have exhibited an infinite family of elements of SPDS in which  $a^2$  appears four times. And finally, we note that an alternate proof can be given using  $m = a \cdot 10^{2n+1} + \frac{a}{2}10^n + a$

$$\text{for which } m^2 = a^2 \cdot 10^{4n+2} + a^2 \cdot 10^{3n+1} + (45a)^2 \cdot 10^{2n-2} + a^2 \cdot 10^n + a^2.$$

This concludes our results emphasizing the infinitude of SPDS. In addition we wish to note an instance of the square of an element of SPDS which also belongs to SPDS, namely  $441^2 = 194481$ .

Can an example be found of integers  $m$ ,  $m^2$ ,  $m^4$  all belonging to SPDS? It is relatively easy to find two consecutive squares in SPDS. One example is  $12^2 = 144$  and  $13^2 = 169$ . Does SPDS also contain a sequence of three or more consecutive squares?

Reference:

[1] Charles Ashbacher, Collection of Problems On Smarandache Notions, Erhus University Press, Vail, 1996.

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# REMARKABLE INEQUALITIES

by

Ion Bălăcenoiu

In this paper are presented inequalities between factors of canonical decomposition.  
Let

$$n! = p_1^{e_{p_1}(n)} \cdot p_2^{e_{p_2}(n)} \cdot \dots \cdot p_{\pi(n)}^{e_{p_{\pi(n)}}(n)}$$

be the decomposition of  $n!$  into primes with  $2 = p_1 < 3 = p_2 < \dots < p_{\pi(n)}$ ,  
and  $\pi(n)$  is the number of prime numbers smaller or equal to  $n$ . Of course,  $e_{p_i}(n)$ ,  $i = \overline{1, \pi(n)}$   
are Legendre's exponents. It is said that:

$$e_p(n) = \left[ \frac{n}{p} \right] + \left[ \frac{n}{p^2} \right] + \left[ \frac{n}{p^3} \right] + \dots$$

## 1. Proposition.

For every  $n \geq 4$ , holds:  $2^{e_2(n)} > 3^{e_3(n)}$

**Proof.** Because

$$\frac{\left[ \frac{n}{2} \right] + \left[ \frac{n}{2^2} \right]}{\left[ \frac{n}{3} \right] + \left[ \frac{n}{3^2} \right]} > \frac{\frac{n}{2} - 1 + \frac{n}{2^2} - 1}{\frac{n}{3} + \frac{n}{3^2}} = \frac{9(3n-8)}{16n} \text{ and } \frac{9(3n-8)}{16n} \geq \frac{5}{3} \text{ for } n \geq 216 \text{ it results that:}$$

$$\frac{\left[ \frac{n}{2} \right] + \left[ \frac{n}{2^2} \right]}{\left[ \frac{n}{3} \right] + \left[ \frac{n}{3^2} \right]} \geq \frac{5}{3} \text{ for } n \geq 216 \quad (1)$$

If  $n = 2k$ , then

$$\frac{\left[ \frac{n}{2} \right]}{\left[ \frac{n}{3} \right]} \geq \frac{\frac{n}{2}}{\frac{n}{3}} = \frac{3}{2}$$

If  $n = 2k+1$ , it results that we have the following possibilities:  $2k+1 = 3m$  or  $2k+1 = 3m+1$  or  $2k+1 = 3m+2$  and consequently  $\frac{k}{m} = \frac{3}{2} - \frac{1}{2m}$  or  $\frac{k}{m} = \frac{3}{2}$  or  $\frac{k}{m} = \frac{3}{2} + \frac{1}{2m}$ .

It results

$$\frac{\left[ \frac{n}{2} \right]}{\left[ \frac{n}{3} \right]} = \frac{k}{m} \geq \frac{3}{2} - \frac{1}{14} = \frac{10}{7}, \text{ that is } \left[ \frac{n}{2} \right] \geq \frac{10}{7} \left[ \frac{n}{3} \right] \text{ for } n \geq 21.$$

While

$$\frac{\left[ \frac{n}{2^2} \right]}{\left[ \frac{n}{3^2} \right]} = \frac{\left[ \frac{1}{2} \left[ \frac{n}{2} \right] \right]}{\left[ \frac{1}{3} \left[ \frac{n}{3} \right] \right]} \geq \frac{\left[ \frac{3}{2} \cdot \frac{10}{7} \cdot \frac{1}{3} \left[ \frac{n}{3} \right] \right]}{\left[ \frac{1}{3} \left[ \frac{n}{3} \right] \right]} \geq \frac{\left[ \frac{15}{7} \right] \left[ \frac{1}{3} \left[ \frac{n}{3} \right] \right]}{\left[ \frac{1}{3} \left[ \frac{n}{3} \right] \right]} = \left[ \frac{15}{7} \right] = 2$$

that is

$$\left[ \frac{n}{2^2} \right] \geq 2 \left[ \frac{n}{3^2} \right].$$

And 
$$\frac{\left\lfloor \frac{n}{2^3} \right\rfloor}{\left\lfloor \frac{n}{3^3} \right\rfloor} = \frac{\left\lfloor \frac{1}{2} \left\lfloor \frac{n}{2^2} \right\rfloor \right\rfloor}{\left\lfloor \frac{1}{3} \left\lfloor \frac{n}{3^2} \right\rfloor \right\rfloor} \geq \frac{\left\lfloor \frac{3}{2} \cdot 2 \cdot \frac{1}{3} \left\lfloor \frac{n}{3^2} \right\rfloor \right\rfloor}{\left\lfloor \frac{1}{3} \left\lfloor \frac{n}{3^2} \right\rfloor \right\rfloor} \geq 3$$

Generally, is true that:

$$\frac{\left\lfloor \frac{n}{2^i} \right\rfloor}{\left\lfloor \frac{n}{3^i} \right\rfloor} \geq \frac{5}{3} \quad \text{for } i \geq 3 \quad (2)$$

From (1) and (2) it results  $\frac{e_2(n)}{e_3(n)} \geq \frac{5}{3}$  for  $n \geq 216$  and so  $2^{\frac{e_2(n)}{e_3(n)}} \geq 2^{\frac{5}{3}} > 3$  or  $2^{e_2(n)} > 2^{e_3(n)}$  for  $n \geq 216$ .

It may be verified directly that  $2^{e_2(n)} > 3^{e_3(n)}$  for  $4 \leq n < 216$ .

## 2. Proposition.

For  $p \geq 5$  and  $n \geq 2$  it is true that  $2^{e_2(n)} > p^{e_p(n)}$ .

**Proof.**

i) If  $2 \leq n < p$  because  $e_p(n) = 0$ , it results  $2^{e_2(n)} > p^{e_p(n)} = 1$

ii) If  $5 \leq p \leq n < p^2$ , then it may be showed that:

$$\frac{p}{2} \leq \frac{\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor}{\left\lfloor \frac{n}{p} \right\rfloor} \leq \frac{e_2(n)}{e_p(n)} \quad (3)$$

iii) For  $n \geq p^2$  we have:

$$\frac{\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor}{\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor} > \frac{\frac{n}{2} - 1 + \frac{n}{2^2} - 1}{\frac{n}{p} + \frac{n}{p^2}} = \frac{p^2(3n-8)}{4n(p+1)} \quad \text{and of course:}$$

$$\frac{p^2(3n-8)}{4n(p+1)} > \frac{p}{2} \Leftrightarrow n > \frac{8p}{p-2}.$$

Therefore  $n \geq p^2 \geq n > \frac{8p}{p-2}$  for  $p \geq 5$ , it results

$$\frac{\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor}{\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor} > \frac{p}{2} \quad (4)$$

If  $n = 2k$ , then:

$$\frac{\left\lfloor \frac{n}{2} \right\rfloor}{\left\lfloor \frac{n}{p} \right\rfloor} > \frac{\frac{n}{2}}{\frac{n}{p}} = \frac{p}{2} > \frac{p-1}{2} \quad (5)$$

If  $n = 2k+1 \geq p$ , then  $2k+1$  is of the form:  $2k+1 = pm+i$ ,  $i \in \overline{0, p-1}$ . For  $i=0$ , it results

$$\frac{k}{m} = \frac{p}{2} - \frac{1}{2m} \geq \frac{p-1}{2} \quad \text{and for } i \in \overline{1, p-1}, \quad \frac{k}{m} = \frac{p}{2} + \frac{i}{2m}.$$

Finally we get

$$\frac{\left\lfloor \frac{n}{2} \right\rfloor}{\left\lfloor \frac{n}{p} \right\rfloor} = \frac{k}{m} \geq \frac{p-1}{2} \quad (5')$$

We have also

$$\frac{\left\lfloor \frac{n}{2^2} \right\rfloor}{\left\lfloor \frac{n}{p^2} \right\rfloor} = \frac{\left\lfloor \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor}{\left\lfloor \frac{1}{p} \left\lfloor \frac{n}{p} \right\rfloor \right\rfloor} \geq \frac{\left\lfloor \frac{p(p-1)}{2} \frac{1}{2} \left\lfloor \frac{n}{p} \right\rfloor \right\rfloor}{\left\lfloor \frac{1}{p} \left\lfloor \frac{n}{p} \right\rfloor \right\rfloor} \geq \frac{\left\lfloor \frac{p(p-1)}{4} \right\rfloor \left\lfloor \frac{1}{p} \left\lfloor \frac{n}{p} \right\rfloor \right\rfloor}{\left\lfloor \frac{1}{p} \left\lfloor \frac{n}{p} \right\rfloor \right\rfloor} = \left\lfloor \frac{p(p-1)}{4} \right\rfloor$$

While

$$\frac{\left\lfloor \frac{n}{2^3} \right\rfloor}{\left\lfloor \frac{n}{p^3} \right\rfloor} \geq \frac{\left\lfloor \frac{p}{2} \left\lfloor \frac{p(p-1)}{4} \right\rfloor \frac{1}{p} \left\lfloor \frac{n}{p^2} \right\rfloor \right\rfloor}{\left\lfloor \frac{1}{p} \left\lfloor \frac{n}{p^2} \right\rfloor \right\rfloor} \geq \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p(p-1)}{4} \right\rfloor$$

Generally

$$\frac{\left\lfloor \frac{n}{2^i} \right\rfloor}{\left\lfloor \frac{n}{p^i} \right\rfloor} \geq \left\lfloor \frac{p}{2} \right\rfloor^{i-2} \left\lfloor \frac{p(p-1)}{4} \right\rfloor \geq \frac{p}{2} \quad \text{for } p \geq 5, \quad i \geq 2 \quad (6)$$

From (4) and (6) it results that

$$\frac{e_2(n)}{e_p(n)} \geq \frac{p}{2} \quad \text{for } n \geq p^2 \quad (7)$$

From (3) and (7) it results  $2^{\frac{e_2(n)}{e_p(n)}} \geq 2^{\frac{p}{2}} > p$ , that is:  $2^{e_2(n)} > 2^{e_p(n)}$

### 3. Proposition.

Let  $p, q$  be prime numbers,  $n = p \cdot q \cdot x$  with  $x \in \mathbb{N}^*$ . If  $3 \leq p < q$  and  $\left\lfloor \frac{q^2}{p^2} \right\rfloor > \frac{q}{p}$ , it results  $p^{e_p(n)} > q^{e_q(n)}$ .

**Proof.** Obviously, if  $n = p \cdot q \cdot x$  then:

$$\frac{\left\lfloor \frac{n}{p} \right\rfloor}{\left\lfloor \frac{n}{q} \right\rfloor} = \frac{qx}{px} = \frac{q}{p} > 1 \quad (8)$$

We shall prove that

$$\frac{e_p(n)}{e_q(n)} = \frac{\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots}{\left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{n}{q^2} \right\rfloor + \left\lfloor \frac{n}{q^3} \right\rfloor + \dots} \geq \frac{q}{p} \quad (9)$$

For  $n = p \cdot q \cdot x$ ,  $x \in \mathbb{N}^*$  is true the following inequality

$$\frac{\left\lfloor \frac{n}{p^i} \right\rfloor}{\left\lfloor \frac{n}{q^i} \right\rfloor} \geq \left\lfloor \frac{q}{p} \right\rfloor^{i-2} \left\lfloor \frac{q^2}{p^2} \right\rfloor \geq \left\lfloor \frac{q^2}{p^2} \right\rfloor, \quad \text{if } p < q, \quad i \geq 2 \quad (10)$$

We prove this inequality using the mathematical induction. Obviously

$$\frac{\left\lfloor \frac{n}{p^2} \right\rfloor}{\left\lfloor \frac{n}{q^2} \right\rfloor} = \frac{\left\lfloor \frac{1}{p} \left\lfloor \frac{n}{p} \right\rfloor \right\rfloor}{\left\lfloor \frac{1}{q} \left\lfloor \frac{n}{q} \right\rfloor \right\rfloor} \geq \frac{\left\lfloor \frac{q^2}{p^2} \frac{1}{q} \left\lfloor \frac{n}{q} \right\rfloor \right\rfloor}{\left\lfloor \frac{1}{q} \left\lfloor \frac{n}{q} \right\rfloor \right\rfloor} \geq \frac{\left\lfloor \frac{q^2}{p^2} \right\rfloor \left\lfloor \frac{1}{q} \left\lfloor \frac{n}{q} \right\rfloor \right\rfloor}{\left\lfloor \frac{1}{q} \left\lfloor \frac{n}{q} \right\rfloor \right\rfloor} = \left\lfloor \frac{q^2}{p^2} \right\rfloor$$

We suppose that is true (10) for  $i=k-1$ , that is:

$$\frac{\left\lfloor \frac{n}{p^{k-1}} \right\rfloor}{\left\lfloor \frac{n}{q^{k-1}} \right\rfloor} \geq \left\lfloor \frac{q}{p} \right\rfloor^{k-3} \left\lfloor \frac{q^2}{p^2} \right\rfloor \geq \left\lfloor \frac{q^2}{p^2} \right\rfloor \quad (10')$$

And we demonstrate that (10) is true for  $i=k$ :

$$\frac{\left\lfloor \frac{n}{p^k} \right\rfloor}{\left\lfloor \frac{n}{q^k} \right\rfloor} = \frac{\left\lfloor \frac{1}{p} \left\lfloor \frac{n}{p^{k-1}} \right\rfloor \right\rfloor}{\left\lfloor \frac{1}{q} \left\lfloor \frac{n}{q^{k-1}} \right\rfloor \right\rfloor} \geq \frac{\left\lfloor \frac{q}{p} \left\lfloor \frac{q}{p} \right\rfloor^{k-3} \left\lfloor \frac{q^2}{p^2} \right\rfloor \frac{1}{q} \left\lfloor \frac{n}{q^{k-1}} \right\rfloor \right\rfloor}{\left\lfloor \frac{1}{q} \left\lfloor \frac{n}{q^{k-1}} \right\rfloor \right\rfloor} \geq \left\lfloor \frac{q}{p} \right\rfloor^{k-2} \left\lfloor \frac{q^2}{p^2} \right\rfloor \geq \left\lfloor \frac{q^2}{p^2} \right\rfloor$$

Finally the formula (10) is true for  $i \geq 2$ .

If  $\left\lfloor \frac{q^2}{p^2} \right\rfloor > \frac{q}{p}$  then from (8) and (10) it results (9). Using (9) it results:

$$\frac{e_p(n)}{p^{e_q(n)}} \geq p^{\frac{q}{p}} \quad (11)$$

Because  $p^q > q^p$  is  $q > p \geq 3$  it results  $p^{\frac{q}{p}} > q$  and therefore  $\frac{e_p(n)}{p^{e_q(n)}} \geq p^{\frac{q}{p}} > q$  that is:

$$p^{e_p(n)} > p^{e_q(n)} \quad (11')$$

#### 4. Remark.

The restriction  $3 \leq p < q$  is suppressed in following cases:

i)  $p = 2$  and  $q \geq 5$ , because in Proposition 2 it is showed that:

$$2^{e_2(n)} > q^{e_q(n)}, \quad \text{for } n \geq 2.$$

ii)  $p = 2$ ,  $q = 3$  for  $n = 2 \cdot 3 \cdot x$ ,  $x \in \mathbb{N}^*$ ,  $6 \mid x$  and  $x \geq 18$ . Is true that:

$$\frac{\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor}{\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{3^2} \right\rfloor} \geq \frac{5}{3}, \quad \text{that is} \quad \frac{3x + \left\lfloor \frac{3x}{2} \right\rfloor}{2x + \left\lfloor \frac{2x}{3} \right\rfloor} \geq \frac{5}{3}.$$

Obviously 
$$\frac{3x + \left\lfloor \frac{3x}{2} \right\rfloor}{2x + \left\lfloor \frac{2x}{3} \right\rfloor} \geq \frac{5}{3} \Leftrightarrow 0 < 3 \left\lfloor \frac{3x}{2} \right\rfloor - 5 \left\lfloor \frac{2x}{3} \right\rfloor - x$$

Because  $3\left(\frac{3x}{2}-1\right)-5\frac{2x}{3}-x \leq 3\left[\frac{3x}{2}\right]-5\left[\frac{2x}{3}\right]-x$ , and  $0 \leq \frac{x-18}{6}$  for  $x \geq 18$ ,  
it results:

$$\frac{\left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right]}{\left[\frac{n}{3}\right] + \left[\frac{n}{3^2}\right]} \geq \frac{5}{3} \quad (12)$$

Using (2) and (12) it results:

$$\frac{e_2(n)}{2e_3(n)} \geq 2\frac{5}{3} > 3 \quad \text{and therefore} \quad 2^{e_2(n)} > 3^{e_3(n)}$$

iii)  $p=2$ ,  $q=3$  and  $n=2^2 \cdot 3^2 \cdot x$ , where  $x \in \mathbb{N}^*$ .

Indeed

$$\frac{\left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right]}{\left[\frac{n}{3}\right] + \left[\frac{n}{3^2}\right]} = \frac{27}{16} > \frac{5}{3} \quad (13)$$

Using (2) and (13) it results:

$$\frac{e_2(n)}{2e_3(n)} \geq 2\frac{5}{3} > 3 \quad \text{and therefore} \quad 2^{e_2(n)} > 3^{e_3(n)}$$

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# Some Considerations Concerning the Sumatory Function Associated to Generalised Smarandache Function

E. Radescu and N. Radescu

Let us denote by  $\overset{d}{V}$  the least common multiple, by  $\overset{d}{\Lambda}$  the greatest common divisor and  $\Lambda = \min, V = \max$ . It is known that  $N_0 = (N^*, \Lambda, V)$  and  $N_d = (N^*, \overset{d}{\Lambda}, \overset{d}{V})$  are lattices. The order on the set  $N^* : n_1 \leq n_2 \Leftrightarrow n_1 \Lambda n_2 = n_1$ , corresponding to the first of these lattices and it is known that this is a total order. But the order  $\leq_d$  induced on the same set by  $\overset{d}{\Lambda}$  and  $\overset{d}{V}$  and defined by:  $n_1 \leq_d n_2 \Leftrightarrow n_1 \overset{d}{\Lambda} n_2 = n_1 \Leftrightarrow n_1$  divides  $n_2$  is only a partial order.

Let  $\sigma : N_0 \rightarrow N_d$  (1) a sequence of positive integers defined on the set  $N^*$ . The sequence (1) is said to be a multiplicatively convergent to zero sequence (mcz) if:  
 $\forall n \in N^*, \exists m_n \in N^*, \forall m > m_n \Rightarrow n \leq_d \sigma(m)$  (2).

The sequence

$$\sigma : N_d \rightarrow N_d \quad (3)$$

is said to be a divisibility sequence (ds) if:  $n \leq_d m \Rightarrow \sigma(n) \leq_d \sigma(m)$  and it is said to be a strong divisibility sequence (sds) if:

$$\sigma\left(n \overset{d}{\Lambda} m\right) = \sigma(n) \overset{d}{\Lambda} \sigma(m) \text{ for every } n, m \in N^* \quad (4).$$

Let the lattices  $N_0$  and  $N_d$ . We'll use the following notations:

- (a) a sequence  $\sigma_{00} : N_0 \rightarrow N_0$  is a (oo) - sequences;
- (b) a sequence  $\sigma_{0d} : N_0 \rightarrow N_d$  is a (od) - sequences;
- (c) a sequence  $\sigma_{d0} : N_d \rightarrow N_0$  is a (do) - sequences;
- (d) a sequence  $\sigma_{dd} : N_d \rightarrow N_d$  is a (dd) - sequences.

Then A(do) - sequence  $\sigma_{d0}$  the monotonicity yields:

$$(m_{d0}) \forall n_1, n_2 \in N^*, n_1 \leq_d n_2 \Rightarrow \sigma_{d0}(n_1) \leq \sigma_{d0}(n_2) \quad (5)$$

and the condition of convergence to infinity is:

$$(c_{d0}) \forall n \in N^*, \exists m_n \in N^*, \forall m \leq_d m_n \Rightarrow \sigma_{d0}(m) \geq n \quad (6).$$

Analogously, for a (dd) - sequence  $\sigma_{dd}$  the monotonicity yields:

$$(m_{dd}) \forall n_1, n_2 \in N^*, n_1 \leq_d n_2 \Rightarrow \sigma_{dd}(n_1) \leq_d \sigma_{dd}(n_2) \quad (7)$$

and the convergence to zero is:

$$(c_{dd}) \forall n \in N^*, \exists m_n \in N^*, \forall m \geq_d m_n \Rightarrow \sigma_{dd}(m_n) \geq_d n. \quad (8)$$

To each sequence  $\sigma_{ij}$ , with  $i, j \in \{0, d\}$ , satisfying the condition  $(c_{ij})$ , one may attach a sequence  $S_{ij}$  (a generalised Smarandache function) defined by:

$$S_{ij} = \min \{m_n : m_n \text{ is given by the condition } (c_{ij})\} \quad (9).$$

For the properties the functions  $S_{ij}$ , see [2].

It is said that for every numerical function  $f$  it can be attached the sumatory function:

$$F_f(n) = \sum_{d|n} f(d) \quad (10)$$

The function  $f$  is expressed as:

$$f(n) = \sum_{v|n} \mu(v) F_f(v) \quad (11)$$

where  $\mu$  is the Mobius function ( $\mu(1) = 1, \mu(n) = 0$  if  $n$  is divisible by the square of a prime number,  $\mu(n) = (-1)^k$  if  $n$  the product of  $k$  different prime numbers).

If  $f$  is the a generalised Smarandache function,  $S_{ij}$  then

$$F_{ij}^s(n) = \sum_{d|n} S_{ij}(d), i, j \in \{0, d\}. \quad (12)$$

Now let us consider  $n = p_1 p_2 \dots p_k$ , with  $p_1 < p_2 < \dots < p_k$  primes number and  $S_{ij}(p_1) \leq S_{ij}(p_2) \leq \dots \leq S_{ij}(p_k)$ , for example. If  $i=0, j=d$ , then  $S_{0d}\left(n_1 \overset{d}{\vee} n_2\right) = S_{0d}(n_1) \vee S_{0d}(n_2)$  and

$$F_{0d}^s(n) = S_{0d}(1) + \sum_{h=1}^k S_{0d}(p_h) + \sum_{h,t=1, h \neq t}^k S_{0d}(p_h p_t) + \sum_{h,t,q=1, h \neq t \neq q}^k S_{0d}(p_h p_t p_q) + \dots S_{0d}(n). \text{ It result:}$$

$$F_{0d}^s(1) = S_{0d}(1);$$

$$F_{0d}^s(p_1) = S_{0d}(1) + S_{0d}(p_1) = F_{0d}^s(1) + 2^0 S_{0d}(p_1);$$

$$F_{0d}^s(p_1 p_2) = S_{0d}(1) + S_{0d}(p_1) + S_{0d}(p_2) + S_{0d}(p_1 p_2) = S_{0d}(1) + S_{0d}(p_1) + 2 S_{0d}(p_2) = F_{0d}^s(p_1) + 2 S_{0d}(p_2);$$

$$F_{0d}^s(p_1 p_2 p_3) = F_{0d}^s(p_1 p_2) + 2^2 S_{0d}(p_3);$$

$$F_{0d}^s(p_1 p_2 p_3 p_4) = F_{0d}^s(p_1 p_2 p_3) + 2^3 S_{0d}(p_4);$$

$$F_{0d}^s(p_1 p_2 \dots p_k) = F_{0d}^s(p_1 p_2 \dots p_{k-1}) + 2^{k-1} S_{0d}(p_k).$$

$$\text{That is } F_{0d}^s(p_1 p_2 \dots p_k) = S_{0d}(1) + \sum_{i=1}^k 2^{i-1} S_{0d}(p_i).$$

The equality (11) becomes:

$$\begin{aligned} S_{0d}(p_k) &= S(n) = \sum_{ab=n} \mu(n) F_{0d}^s(b) = \\ &= F_{0d}^s(n) - \sum_i F_{0d}^s\left(\frac{n}{p_i}\right) + \sum_{i \neq j} F_{0d}^s\left(\frac{n}{p_i p_j}\right) + \dots \end{aligned}$$

$$\begin{aligned} \text{with } F_{0d}^s\left(\frac{n}{p_i}\right) &= F_{0d}^s(p_1 p_2 \dots p_{i-1} p_{i+1} \dots p_k) = \sum_{j=1}^{i-1} 2^{j-1} S_{0d}(p_j) + \sum_{j=i+1}^k 2^{j-1} S_{0d}(p_j) = \\ &= F_{0d}^s(p_1 p_2 \dots p_{i-1}) + 2^{i-1} F_{0d}^s(p_{i+1} \dots p_k). \end{aligned}$$

Analogously,

$$\begin{aligned} F_{0d}^s\left(\frac{n}{p_i p_j}\right) &= F_{0d}^s(p_1 \dots p_{i-1}) + 2^{i-1} F_{0d}^s(p_{i+1} \dots p_{j-1}) + 2^{j-1} F_{0d}^s(p_{j+1} \dots p_k) = \\ &= \sum_{h=1}^{i-1} 2^{h-1} S_{0d}(p_h) + \sum_{h=i+1}^{j-1} 2^{h-2} S_{0d}(p_h) + \sum_{h=j+1}^k 2^{h-3} S_{0d}(p_h). \end{aligned}$$

In particular, for  $n = p^a$ ,  $p$  prime number, it result:

$$S_{0d}(p^a) = \sum_{a+q=a} \mu(p^q) F_{0d}^s(p^q) = F_{0d}^s(p^a) - F_{0d}^s(p^{a-1}).$$

If  $n = p^a q^b$  with  $\max \{S_{0d}(p), \dots, S_{0d}(p^a)\} \leq \min \{S_{0d}(q), \dots, S_{0d}(q^b)\}$ , then

$$F_{0d}^s(p^a q^b) = F_{0d}^s(p^a) + (a+1) F_{0d}^s(q^b).$$

If  $i=d, j=d$  and if  $\sigma_{dd}$  is a (sds) satisfying the condition  $(c_{dd})$ , then

$$S_{dd}\left(n_1 \overset{d}{\vee} n_2\right) = S_{dd}(n_1) \overset{d}{\vee} S_{dd}(n_2) \quad (13)$$

$$\begin{aligned} \text{and } F_{dd}^s(n) &= S_{dd}(1) + \sum_{h=1}^k S_{dd}(p_h) + \sum_{h,t=1, h \neq t}^k \left[ S_{dd}(p_h) \overset{d}{\vee} S_{dd}(p_t) \right] + \\ &+ \sum_{h,t,q=1, h \neq t \neq q}^k \left[ S_{dd}(p_h) \overset{d}{\vee} S_{dd}(p_t) \overset{d}{\vee} S_{dd}(p_q) \right] + \dots + S_{dd}(n) \end{aligned} \quad (14)$$

$$S_{dd}(p^a) + F_{dd}^s(p^a) - F_{dd}^s(p^{a-1}). \quad (15)$$

Example: The Fibonacci sequence  $(F_n)_{n \in \mathbb{N}^+}$  defined by  $F_{n+1} = F_n + F_{n-1}$ , with  $F_1 = F_2 = 1$  is a (sds), so for the generalised Smarandache function  $S_F$  attached to this sequence we have:

$S_F\left(n_1 \overset{d}{\vee} n_2\right) = S_F(n_1) \overset{d}{\vee} S_F(n_2)$ , and the calculus of  $S_p(n)$  is reduced to the calculus of  $S_F(p^a)$ ,

with  $p$  a prime number. For instance:

n	$S_F(n)$	n	$S_F(n)$	n	$S_F(n)$
1	1	7	8	13	21
2	3	8	6	14	24
3	4	9	24	15	20
4	6	10	15	16	12
5	5	11	20	17	
6	12	12	12	18	



$$F_{dd}^S(4) = 10, F_{dd}^S(8) = 16, F_{dd}^S(16) = 28, F_{dd}^S(15) = 30.$$

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# SuperCommuting and a second distributive law:

## Subtraction and division may not commute, but they SuperCommute

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### Abstract

This paper deals with teaching methods.

Elementary textbooks tell that addition and multiplication commute but subtraction and division do not. Actually they do if a simple restriction is observed. The technique is not new, but the method presented here for teaching it is believed to be new and simple enough for presentation immediately following the signed-numbers concept. The technique is dubbed *SuperCommuting* or the *shuffling property*.

SuperCommuting leads directly to a new formal algebraic distributive law, one that applies to expressions of the form  $1/(a*b/c/d*e\dots)$ . Also, by comparison with the first distributive law, the *duality* concept can be painlessly and unobtrusively introduced by the dedicated instructor of beginning algebra.

### Introduction

The beginning algebra student is today told, almost incidentally, that a long string of numbers consisting of a mixture of additions and subtractions can be evaluated by adding all positive numbers then adding all negative numbers then subtracting the two results. This paper gives a formal treatment of that property and extends it to a similar procedure with strings of multiplications and divisions. Underlying this entire discussion is the *Order of Operations* rule in which operations of a given kind are to be performed from left to right.

Asterisk is used as the multiplication sign to prepare the student for future computer math literacy. Thus  $A*B$  is always used, never  $AB$ ,  $A \times B$ , or  $A \cdot B$ . Similarly, division is always  $C/D$  never  $C \div D$ .

### Subtraction and SuperCommuting

Today, the student may be told that  $A-B$  can be written  $-B+A$ ; but how does one initially present this idea with total clarity? One method follows.

First write  $A-B$  as  $0 + A - B$  where "A" has a plus sign directly attached to it, and "B" has a minus sign directly attached to it. The student would then be told it's alright to shuffle these numbers if the sign of each number is carried with it while keeping 0 at the front of the string. Thus,  $0 + A - B$  becomes  $0 - B + A$  or simply  $-B + A$ , the initial 0 having outlived its usefulness; and the problem in subtraction is said to be the addition of signed numbers.

That simple and obvious development suggests another one that is just as simple but perhaps less obvious.

## Division and SuperCommuting

A similar process can be illustrated with division. Begin by writing  $C/D$  as  $1 * C / D$ , where "C" has a multiplication sign directly attached to it and "D" has a division sign directly attached to it just as if  $*C$  and  $/D$  were some kind of "signed numbers"! The student would then be told it's alright to shuffle these if the sign attached to each number is carried with it, and if 1 is kept at the front of the string. Thus  $1 * C / D$  becomes  $1 / D * C$ , and a problem which started out as division is now seen to be multiplication by the divisor's reciprocal. In this case the initial numeral, 1, must be retained because  $*C$  and  $/D$  are not recognized as any kind of signed numbers. (But might they be?)

## A second distributive law

The student would now be informed that long strings of additions and subtractions can be similarly shuffled into an arbitrary order, as can long strings of multiplications and divisions; and that the traditional commutative laws of addition and multiplication are simply narrow applications of this *shuffling property*.

The parallel, or *dual* nature of the above two developments is as obvious as the fact that lightning begets thunder. Thus is suggested a new distributive law, one that works for complex fractions like  $1/(a*b/c)$  by considering their dual, in this case  $0-(a+b-c)$ . The suggestion is that  $1/(a*b/c)$  becomes  $1/a/b*c$ . A traditional proof is left to the reader.

## Teaching duality early

After this, the student of elementary algebra should be perfectly comfortable with the duality concept if presented something as follows.

One's left hand is like one's right hand except that they are *mutual mirror images*, each is a mirror image or a *reflection* of the other. This is one kind of symmetry. Another kind of symmetry exists between a number and its reciprocal. A number and its reciprocal are *mutual reciprocals*. Still another kind of symmetry is the way in which the new distributive law relates to the old one.

Compare these two forms; the first illustrates the new distributive law, the second form illustrates the old familiar one:

$$1 / (a * b / c) = 1 / a / b * c$$

$$0 - (a + b - c) = 0 - a - b + c$$

The student can then be told that either form can be changed into the other by exchanging  $*$  and  $+$  signs,  $/$  and  $-$  signs, and constants 1 and 0; and that this kind of symmetry is called *duality*, and the two forms are said to be *mutual duals*. It might also be suggested that this kind of duality is a precise form of the usually imprecise method called *analogy*.

Finally it should be pointed out to the student that in each of the three kinds of symmetry discussed above, a double application is the same as no application. That is, if a right hand is reflected twice, the result has the shape of a right hand; if the reciprocal is taken twice, the original number results; and if the dual is taken twice, the original form results.

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# PROPERTIES OF THE TRIPLETS $p^*$

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For every natural number  $p$  we define  $p^*$  as the following triplet  $(p^* - 1, p^*, p^* + 1)$ , where  $p^* = 2.3.5 \dots p$

Let us consider the following sequence of prime numbers :

$$2 = p_1 < 3 = p_2 < 5 = p_3 < \dots < p_k < \dots$$

We call the triplets  $(p_k^* - 1, p_k^*, p_k^* + 1)$ , where  $p_k^* = p_1 \cdot p_2 \cdot \dots \cdot p_k$ ,  $k = 1, 2, \dots$  as  $p^*$  triplets.

It is easy to observe that :

i)  $(p_k^* - 1, p_k^* + 1) = 1$ , because  $p_k^* - 1, p_k^* + 1$  are both of them are odd numbers, and  $(p_k^* + 1) - (p_k^* - 1) = 2$

ii) if  $n = s_1^{\alpha_1} \cdot s_2^{\alpha_2} \cdot \dots \cdot s_t^{\alpha_t}$  divides  $p_k^* - 1$  or  $p_k^* + 1$ , because  $(p_k^* - 1, p_k^*) = 1$ , this implies  $s_i > p_k$ , for every  $i \in \overline{1, t}$ .

iii) if  $n$  divides  $p_k^* - 1$  or  $p_k^* + 1$ , then  $(n, p_h) = 1$ , for  $h \leq k$

Proposition. The triplets  $p^*$  are separated.

Proof. Let us consider the consecutive triplets :

$$p_{k-1}^* - 1, p_{k-1}^*, p_{k-1}^* + 1$$

$$p_k^* - 1, p_k^*, p_k^* + 1$$

Because  $p_k^* - 1 - (p_{k-1}^* + 1) = p_{k-1}^* (p_k - 1) - 2 > 0$  it results that every two consecutive triplets are separated, so we have :

$$p_1^* - 1 < p_1^* < p_1^* + 1 < p_2^* - 1, p_2^* < p_2^* + 1 < \dots < p_k^* - 1 < p_k^* < p_k^* + 1 < \dots$$

Remark. Let us consider the triplets :

$$p_k^* - 1, p_k^*, p_k^* + 1$$

$$p_h^* - 1, p_h^*, p_h^* + 1, \text{ where } k < h, \text{ and}$$

$$M_{kh} = \{n \in N / p_k^* + 1 < n < p_h^* - 1\}$$

Then we have :

a) if  $h - k$  is constant, then card  $M$  increases simultaneously with  $k$ .

b) card  $M_{kh}$  increases when  $h - k$  increases.

Definition. We say that the triplets  $p_k^*, p_h^*$ , where  $k < h$ , are F - prime triplets iff there is no  $n \in N, n > 1$  so that  $n / p_k^* \pm 1$  and  $n / p_h^* \pm 1$  or  $n / p_h^*$  or  $n / p_k^* \pm 1$

Examples. The triplets :

$$5^* - 1 = 29, 5^* = 30, 5^* + 1 = 31$$

$$7^* - 1 = 209, 7^* = 210, 7^* + 1 = 211 \text{ are}$$

F - prime triplets.

The triplets :

$$7^* - 1 = 209, 7^* = 210, 7^* + 1 = 211$$

$$11^* - 1 = 2309, 11^* = 2310, 11^* + 1 = 2311$$

are not F - prime triplets, because  $(7^* - 1, 11^*) = 11$

Definition . The triplets :  $(p^* - 1, p^*, p^* + 1)$  and  $(q^* - 1, q^*, q^* + 1)$  where  $p^* - 1 = q$  or  $p^* + 1 = q$  are called tinked triplets.

Remark. i) If  $q$  and  $p$  are two consecutive prime numbers, then we call  $p^*$  and  $q^*$  as consecutive linked triplets. For example  $3^*$  and  $5^*$  are consecutive linked triplets.

ii) Two linked triplets are not F - prime triplets.

Proposition . There is no consecutive linked triplets with  $p < q$ , for every  $p \geq 5$ .

Proof. Because  $p$  and  $q$ ,  $p < q$ , are two consecutive prime numbers, we have :  $p < q < 2p$ .

For every  $p \geq 5$  we have :

$$\left[ \frac{p^* + 1}{q} \right] = \left[ \frac{p^*}{q} + \frac{1}{q} \right] \geq \left[ \frac{p^*}{2p} + \frac{1}{q} \right] = \left[ \frac{s^*}{2} + \frac{1}{q} \right] = \frac{s^*}{2} \geq 3,$$

where  $s$  is such that  $s < p$  and  $s$  and  $p$  are two consecutive prime number, so we have :  $p^* + 1 \neq q$ .

Because  $\left[ \frac{p^* - 1}{q} \right] \geq \left[ \frac{s^*}{2} - \frac{1}{q} \right] = \frac{s^*}{2} - 1 \geq 2$ , then we have  $p^* - 1 \neq q$

Remark i) There are  $p^*$  triplets such that  $p^* - 1$  and  $p^* + 1$  are friend prime numbers ( for example for  $p = 5$  )

There are friend prime numbers which do not belong to a  $p^*$  triplet . For example the friend prime number 11 and 13 do not belong to any triplet  $p^*$ , because 12 is not a  $p^*$ .

ii) The friend prime numbers which belong to a triplet  $p^*$  are called friend prime numbers with the triplet  $p^*$ .

There are the pairs of friend prime numbers (5,7) and (29,31) with the triplet  $p^*$  which correspond to  $p^*$  linked consecutive triplets.

Unsolved problem

i) There are an infinite set of friend prime numbers which the triplet  $p^*$ .

ii) There are an infinite set of friend prime numbers which the triplet is not  $p^*$ .

Proposition. For every  $k \in N^*$  there is a natural number  $h, h > k$  such that for every  $s \geq h$ , the triplets  $(p_k^* - 1, p_k^*, p_k^* + 1)$  and  $(p_s^* - 1, p_s^*, p_s^* + 1)$  are not F - prime.

Proof. If  $n$  divides  $p_k^*$  or  $p_k^* + 1$ , then  $n = t_1^{\alpha_1} \dots t_i^{\alpha_i}$ , where  $t_j > p_k$  for every  $j \in \overline{1, i}$ .

Let  $\bar{n}$  be  $\bar{n} = t_1 \cdot t_2 \dots t_i$ .

Then  $\bar{n}$  divides  $p_k^* - 1$  or  $p_k^* + 1$ . If  $p_h = \max\{t_j\}$ , then  $h > k, \bar{n}$  divides  $p_k^*$  and, of course,  $\bar{n}$  divides  $p_s^*$ , for every  $s \geq h$ . Then the triplets  $p_k^*, p_s^*$  are not F - prime.

Definition. If  $n = p_{i_1}^{\alpha_1} \dots p_{i_r}^{\alpha_r}$ , then  $\bar{n}$  is denoted by  $\bar{n} = \{p_{i_1}^{\alpha_1}, p_{i_2}^{\alpha_2}, \dots, p_{i_r}^{\alpha_r}\}$ .

Definition. Let us consider  $M = \{\bar{n}\}_{n \in M}$  and let  $\preceq$  be the partial ordering relation on  $M$ , defined by

$\{p_{i_1}^{\alpha_1}, \dots, p_{i_r}^{\alpha_r}\} \lesssim \{q_{j_1}^{\beta_1}, \dots, q_{j_l}^{\beta_l}\} \Leftrightarrow \{p_{i_1}, p_{i_2}, \dots, p_{i_r}\} \subset \{q_{j_1}, \dots, q_{j_l}\}$  and  $p_{i_k} = q_{j_l}$  implies  $\alpha_k \leq \beta_l$ .

Definition. Let us consider  $M_k = \bar{p}_k^* \cup \bar{p}_k^* - 1 \cup \bar{p}_k^* + 1, k \in N^*$ .

Then we define  $\hat{p}_k^* = \{n \in N^* / \bar{n} \subset M_k\}$  and  $\lesssim$ , for  $h < k$ .

Remark. For  $n = t_1^{r_1} \dots t_i^{r_i}$ , if  $n \in \bar{p}_k^*$ , then there are the following cases :

i)  $n / p_k^*$  and

ii)  $n / p_k^* - 1$  or  $n / p_k^* + 1$

In the first case,  $n \in \{p_k, p_1 p_k, \dots, p_{k-1} p_k, \dots, p_k^*\}$ .

In the second case, because  $t_j > p_k$  for every  $j \in \overline{1, i}$  it implies that there is  $s \in \overline{1, i}$  for every  $h, 1 \leq h \leq k$ , such that  $t_s^{\alpha_s} \notin p_h^* - 1$ , respectively  $t_s^{r_s} \notin p_h^* + 1$ .

In the paper [1] it is defined the Primorial Smarandache function, denoted by  $SP_r$ , where  $SP_r: A \subset N^* \rightarrow N^*$  and  $SP_r(n) = p$ , where  $p$  is the smallest prime number such that  $n$  divides one of the numbers which belong to the triplet  $p^*: p^* - 1, p^*, p^* + 1$ , where  $p^* = 2.3.5. \dots . p$  ( the product of the prime numbers which are  $\leq p$  )

In the paper [1] it is proved that the free of quadrates numbers belongs to the domain of definition of the function  $SP_r$ . The problem is : There are numbers which are not free of quadrates numbers which belongs to the domain of definition of the function  $SP_r$  ?

We study if there is  $x^2 \in N^*$ , where  $x$  is a prime number, such that  $x^2$  divides one of the numbers of the triplet  $p^*: p^* - 1, p^*, p^* + 1$ , where  $p$  is a prime number.

It is easy to see that  $x^2 \nmid p^*$ , for every prime number  $p$ .

We proof that every prime number  $x \in N^*$  has the property  $x^2 \nmid p^* \pm 1$ . If  $x < p$ , then  $x^2 \nmid p^* \pm 1$ .

Proposition.  $x^2 \nmid p^* \pm 1$

Proof. In the case  $x^2 = p^* + 1$ , then  $x^2 - 1 = p^*$ . It is easy to see that  $x = 2$  do not verify this property.

Because  $x^2 - 1 = M4$  and  $p^* = M4 + 2$ , then  $x^2 - 1 \neq p^*$

If  $x = p^* - 1, x^2 + 1 \neq M3$  and  $p^* = M3$ , then  $x^2 + 1 \neq p^*$

Remark. Every free of quadrates number could be of one of the following kinds :  $4kx^2, (4k+1)x^2, (4k+2)x^2$  or  $(4k+3)x^2$ , where  $k \in N$  and  $x$  is a prime number.

Proposition. For every prime number  $x, x \in N$ , we have :

a)  $4kx^2 \neq p^* \pm 1$

b)  $(4k+2)x^2 \neq p^* \pm 1$

c)  $(4k+1)x^2 \neq p^* + 1$

d)  $(4k+3)x^2 \neq p^* - 1$

Proof. a) Because  $4kx^2$  is an even number and  $p^* \pm 1$  are odd numbers, then it results that  $4kx^2 \neq p^* \pm 1$

b) In an analogue way  $(4k+2)x^2 \neq p^* \pm 1$ , because  $(4k+2)x^2$  is an even number.

c) Because  $(4k+1)x^2 - 1 = M4, x > 2$  and  $p^* = M4 + 2$  then it results that  $(4k+1)x^2 \neq p^* + 1$ . For  $x = 2$  it can be directly proved.

d) Because  $(4k+3)x^2+1=M4$ , then it implies  $(4k+3)x^2 \neq p^*-1$ . For  $x=2$  it is directly proved.

In order to prove the proposed problem it is necessary to study the following cases, too:

$\exists x$  and  $p$  which are prime numbers, so that :

a)  $(4k+1)x^2 = p^* - 1$ , where  $4k+1, 4k+3$  are prime number greater than  $x$ .

b)  $(4k+3)x^2 = p^* + 1$  or products of primes greater than  $x$ .

It is easy to see that in the case when  $4k+1$  and  $4k+3$  have a prime factor  $q$  smallest than  $p$  ( $q \leq p$ ) the assertions a) and b) are not proved.

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# ON THE QUATERNARY QUADRATIC DIOPHANTINE EQUATIONS

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In this paper are presented the parametric solutions for the homogeneous diophantine equations:

$$x^2 + by^2 + cz^2 = w^2 \quad (1)$$

where  $b, c$  are rational integers.

I. Present theory. Case 1:  $b = c = 1$

Curmichael [2] showed that the solutions are expresions with the form:

$$\begin{aligned} w &= p^2 + q^2 + u^2 + v^2; y = 2pq + 2uv; \\ x &= p^2 - q^2 + u^2 - v^2; z = 2pv - 2qu; \end{aligned} \quad (2)$$

where  $p, q, u, v$  are rational integers.

Mordell [3] showed that only these are the equations solution's by appying the arithmetic theory of the Gaussian integers.

Case 2:  $b = 1; c = -1$ . Mordell [3] showed that the solutions are, and only these, the expressions:

$$\begin{aligned} 2x &= ad - bc; 2y = ac + bd; \\ 2z &= ac - bd; 2w = ad + bc; \end{aligned} \quad (3)$$

$a, b, c, d$  are integer parameters.

Case 3:  $b, c$ , are rational integers.

Mordell [3] took the particulary solutions with trhee parameters again, had been proposed by Euler:

$$\begin{aligned} w &= p^2 + bq^2 + cu^2; y = 2pq; \\ x &= p^2 - bq^2 - cu^2; z = 2pu; \end{aligned} \quad (4)$$

II. Results.

In [4] is proposed a new method to solve the quaternary equations using the notion of "quadratic combination". If we noted  $G_2^2$ , the complete system of equation's solutions:  $x^2 + y^2 = z^2$ , and also  $G_3^2$  for the equation:  $x^2 + y^2 + z^2 = w^2$ , we sholl can to enuneiate:

Definition 1: Quadratic combination is a numerical function  $\square$  which associates each two solutios from  $G_2^2$ , four solutions from  $G_3^2$ . Simbolically we have:

$$\square : G_2^2 \times G_2^2 \rightarrow G_3^2$$



### Observation.

From the quadratic combination of the equation's solutions with the form:  $x^2 + by^2 = z^2$ , we shall obtain the solutions for the equations  $x^2 + by^2 + cz^2 = w^2$  [4]

#### 1. Case $b = c = 1$

From the quadratic combination, we find again the solution (2). We can present another demonstration for Mordell's sentence. From [4] we have:

##### Theurema 1.

For the equation  $E_3^2$ , the solutions are expresions (2) and only these. The first part of the demonstration results by verification. For the ssecond part of it, we can use the property demonstrated in [4].

**Lemma 2.** The multitude of the equation's solutions  $E_3^2$  is a graph  $F_3^2$  as terminal top the ordinary solution (1, 0, 0, 1) and the arcs are given by the "t" functions:

$$t = w \pm x \pm y \pm z$$

The solutions are matriceally developed:

$$S_{i+1} = S_i \cdot B, \text{ with } B = \begin{pmatrix} 0 & -1 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & -1 & -1 & 2 \end{pmatrix} \quad (5)$$

**Lemma 3.** Any solutions from the equations (2) are on the graph  $F_3^2$  and, reciprocally, any solutions from the  $F_3^2$  can be written with form (2).

It was defined the term:  $t_1 = x + y + z - w$ ;  $t_{i+1} < t_i$ , where variables are naturale numbers [4].

We are verifying that form every solution of naturale numbers can derive a solution whit  $w_1 < w$ .

The parameter's correspondence ( $p > q$  and  $u > v$ ) will be:

$$\begin{aligned} p_1 &= p - q - v; & u_1 &= u + q - v; \\ q_1 &= q; & v_1 &= v \end{aligned}$$

It is obtainid a number of decreasing values  $w_1$ , having as limet the ordinary solutions (1, 0, 0, 1). Reciprocally, for every solution from the graph  $F_3^2$  is obtained a number of parameterly solutions with  $w_1$  breeder, in cas  $t_{i+1} > t_i$ .

#### 2. Case $b = 1, c = -1$ .

From quadratic combination resultes equations:

$$\begin{aligned} w &= p^2 + q^2 - u^2 - v^2 \\ x &= p^2 - q^2 + u^2 - v^2 \\ y &= 2pq + 2uv \\ z &= 2pv + 2qu \end{aligned} \quad (6)$$

It can be showed that the Mordell's solutions (3) are equivalent with solutions (6); the parameter's equivalence is given by:

$$a = p + v; \quad b = p - v$$

$$c = q - u \quad ; \quad d = q + u$$

3. *Case* b, c are rationale integers. For simplicity, we shell treat in two subcases:

3a) b, c prime numbers. The quadratic combination will require the solutions:

$$\begin{aligned} w &= p^2 + bq^2 + cu^2 + bcv^2 \\ x &= p^2 - bq^2 - cu^2 + bcv^2 \\ y &= 2pq + 2cuv \\ z &= 2pu - 2bqv \end{aligned} \quad (7)$$

3b) b and c are compound numbers. For any decomposition:  $b = i \bullet j$  and  $c = l \bullet h$ , where i, j, l, h are rationale integers, we have the general solutions with four parameters of the equation (1):

$$\begin{aligned} w &= ihp^2 + jhq^2 + jlu^2 + ilv^2 \\ x &= ihp^2 - jhq^2 + jlu^2 - ilv^2 \\ y &= 2hpq + 2luv \\ z &= 2ipv - 2jqu \end{aligned} \quad (8)$$

III. Applications We shall take again from [4] only the application of the numerical representations of exponent 2. It is well known the Fermont - Lagrange Theory.

#### *Theorema 2*

For any natural number it is at least a representation by sum of four whole number's square rest:

$$z = u^2 + v^2 + w^2 + t^2 \quad (9)$$

Later on another Theory was demonstrated:

#### *Theorema 3*

For any natural number  $z \neq 2^{2k}(8l+7)$  it is least a representation of three whole a numbers:

$$z = u^2 + v^2 + w^2 \quad (9')$$

Our theory allows us to enunciate a much stranger theory:

#### *Theorema 4*

For any natural number z it is at least three whole numbers (u, v, w) or (a, b, c), in order to have :

$$z = u^2 + v^2 + w^2 \quad (\alpha) \quad (10)$$

$$z = a^2 + b^2 + 2c^2 \quad (\beta)$$

For  $z = z_1 = 2^{2k}(8l+7)$ , we have only the representation (  $\beta$  ), for  $z = z_2 = 2^{2k+1}(8l+7)$ , we have only the representation (  $\alpha$  ) and for  $z \neq z_1 \neq z_2$ , we have in the same time the representations (  $\alpha$  ) and (  $\beta$  ).

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# THE SEMILATTICE WITH CONSISTENT RETURN

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Let  $p$  be a prime number. In [5] is defined the function  $S_p$  as  $S_p: N^* \rightarrow N^*$ ,  $S_p(a) = k$ , where  $k$  is the smallest positive integer so that  $p^a$  is a divisor for  $k!$ .

A Smarandache function of first kind is defined for each  $n \in N^*$  in [1], as numerical function  $S_n: N^* \rightarrow N^*$ , so that:

i) if  $n = u^i$ , where  $u = 1$  or  $u = p$ , then  $S_n(a) = k$ ,  $k$  being the smallest positive integer with the property that  $k! = M \cdot u^{ia}$ .

ii) if  $n = p_1^{i_1} \cdot p_2^{i_2} \cdot \dots \cdot p_r^{i_r}$ , then 
$$S_n(a) = \max_{1 \leq j \leq r} \{S_{p_j}(i_j a)\}.$$

It is proved that:

$$\sum_1 \max\{S_n(a), S_n(b)\} \leq S_n(a+b) \leq S_n(a) + S_n(b)$$

$$\sum_2 S_n(a+b) \leq S_n(a) \cdot S_n(b)$$

In [2] is proved that:

i) the function  $S_n$  is monotonously increasing,

ii) the sequence of functions  $\{S_{p^i}\}_{i \in N^*}$  is monotonously increasing.

iii) for  $p, q$  - prime numbers such that:  $p < q \Rightarrow S_p < S_q$  and  $p \cdot i < q \Rightarrow S_{p^i} < S_{q^j}$ , where  $i \in N^*$

iv) if  $n < p$ , then  $S_n < S_p$ .

In [3] it is proved:

i) for  $p \geq 5$ ,  $S_p > \max\{S_{p-1}, S_{p+1}\}$

ii) for  $p, q$  - prime numbers,  $i, j \in N^*$

$$p < q \text{ and } i \leq j \Rightarrow S_{p^i} < S_{q^j}$$

iii) the sequence of functions  $\{S_n\}_{n \in N^*}$  is generally increasing boundled

iv) if  $n = p_1^{i_1} \cdot p_2^{i_2} \cdot \dots \cdot p_r^{i_r}$ , there are  $k_1, k_2, \dots, k_m \in \{1, 2, \dots, r\}$  so that for each  $t \in \overline{1, m}$  there is  $q_t \in N^*$  so that

$$S_n(q_t) = S_{p_{k_t}^{i_{k_t}}}(q_t)$$

and for each  $l \in N^*$  we have:

$$S_n(l) = \max_{1 \leq t \leq m} \left\{ S_{p_{k_t}^{i_{k_t}}} (l) \right\}.$$

We define the set  $\left\{ p_{k_t}^{i_{k_t}} \mid t \in \overline{1, m} \right\}$  as the set of active factors of  $n$  and the others factors as the pasive factors.

Let  $N_{p_1 \cdot p_2 \cdots p_r} = \left\{ n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r} \mid i_1, i_2, \dots, i_r \in N^* \right\}$ , where  $p_1 < p_2 < \cdots < p_r$  are prime numbers.

Then

$$N^{p_1 p_2 \cdots p_r} = \left\{ n \in N_{p_1 p_2 \cdots p_r} \mid n \text{ has } p_1^{i_1}, p_2^{i_2}, \dots, p_r^{i_r} \text{ as active factors} \right\}$$

is the S-active cone.

A Smarandache function of second kind is defined for each  $k \in N^*$  in [1], as the function  $S^k: N^* \rightarrow N^*$  where  $S^k(n) = S_n(k)$ .

It is proved that:

$$\sum_3 \quad \max \left\{ S^k(a), S^k(b) \right\} \leq S^k(a \cdot b) \leq S^k(a) + S^k(b)$$

$$\sum_4 \quad S^k(a \cdot b) \leq S^k(a) \cdot S^k(b)$$

In [4] it is proved that:

- i) for  $k, n \in N^*$  the formula  $S^k(n) \leq n \cdot k$  is true
- ii) all prime numbers  $p \geq 5$  are maximal points for  $S^k$  and

$$S^k(p) = p \left[ k - i_p(k) \right], \quad \text{where } 0 \leq i_p(k) \leq \left[ \frac{k-1}{p} \right]$$

iii) the function  $S^k$  has its relative minimum values for every  $n = p!$ , where  $p$  is a prime number and  $p \geq \max\{3, k\}$

iv) the numbers  $kp$  for  $p$  prime number,  $k \in N^*$  and  $p > k$ , are the fixed points of  $S^k$

v) the function  $S^k$  have the following properties:

$$a) S^k = 0 \quad (n^{1+\varepsilon}), \text{ for } \varepsilon > 0$$

$$b) \lim_{n \rightarrow \infty} \sup \frac{S^k(n)}{n} = k$$

c)  $S^k$  is, "generally speaking", incresing, thus:

$$\forall n \in N^*, \exists m_0 \in N \text{ so that } \forall m \geq m_0 \Rightarrow S^k(m) \geq S^k(n)$$

1. **DEFINITION.** Let  $\mathcal{M} = \{S_m(n) \mid n, m \in N^*\}$ , let  $A, B \in \mathcal{P}(N^*) \setminus \emptyset$  and  $a = \min A$ ,  $b = \min B$ ,  $a^* = \max A$ ,  $b^* = \max B$ . The set  $I$  is the set of the functions:

$$I_A^B: N^* \rightarrow \mathcal{M}, \text{ with } I_A^B(n) = \begin{cases} S_a(b), n < \max\{a, b\} \\ S_{a_k}(b_k), \max\{a, b\} \leq n \leq \max\{a^*, b^*\} \\ S_{a^*}(b^*), n > \max\{a^*, b^*\} \end{cases}$$

where

$$a_k = \max_i \{a_i \in A \mid a_i \leq n\}$$

$$b_k = \max_j \{b_j \in B \mid b_j \leq n\}$$

## 2. EXAMPLES.

a)  $I_{\{3,8,10\}}^{\{6,10,12\}}: N^* \rightarrow \mathcal{M}$  and:

n	1	2	3	4	5	6	7	8	9	10	11	12	$n \geq 13$
$I_{\{3,8,10\}}^{\{6,10,12\}}$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_8(6)$	$S_8(6)$	$S_{10}(10)$	$S_{10}(10)$	$S_{10}(12)$	$S_{10}(12)$

b) Let  $A = \{1, 3, 5, \dots, 2k+1, \dots\}$

$$B = \{2, 4, 6, \dots, 2k, \dots\}$$

$I_A^B: N^* \rightarrow \mathcal{M}$  and:

n	1	2	3	4	5	6	...	2k	2k+1	...
$I_A^B$	$S_2(1)$	$S_2(1)$	$S_2(3)$	$S_4(3)$	$S_4(5)$	$S_6(5)$	...	$S_{2k}(2k-1)$	$S_{2k}(2k+1)$	...

c) Let  $A = \{5, 9, 10\}$  and  $I_A^A, I_N^{N^*}: N^* \rightarrow \mathcal{M}$  with

n	1	2	3	4	5	6	7	8	9	10	$n \geq 11$
$I_A^A$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_9(9)$	$S_{10}(10)$	$S_{10}(10)$
$I_{N^*}^{N^*}$	$S_1(1)$	$S_2(2)$	$S_3(3)$	$S_4(4)$	$S_5(5)$	$S_6(6)$	$S_7(7)$	$S_8(8)$	$S_9(9)$	$S_{10}(10)$	$S_n(n)$

It is easy to see that  $I_A^A$  is not the reduction of  $I_{N^*}^{N^*}$  and  $I_A^A(N^*) \subset I_{N^*}^{N^*}(N^*)$ .

### 3. REMARK.

The functions which belongs to the set  $I$  have the following properties :

1) if  $A_1 \subset A_2$  and  $n \in A_1$ , then  $I_{A_1}^B(n) = I_{A_2}^B(n)$

1') if  $B_1 \subset B_2$  and  $n \in B_1$ , then  $I_A^{B_1}(n) = I_A^{B_2}(n)$

2)  $I_{N^*}^{N^*}(n) = S_n(n) = S^n(n)$ , the function  $I_{N^*}^{N^*}$  is called the  $I$ -diagonal function and  $I_{N^*}^{N^*}(N^*)$  is called the diagonal of  $\mathcal{M}$ .

3) for each  $m \in N^*$   $I_{\{m\}}^{N^*} = S_m$  for  $I_{\{m\}}^{N^*}(n) = S_m(n), \forall n \in N^*$ .

3') for each  $m \in N^*$   $I_{N^*}^{\{m\}} = S^m$  for  $I_{N^*}^{\{m\}}(n) = S_n(m) = S^m(n), \forall n \in N^*$ ,

4) if  $n \in A \cap B$ , then  $I_A^B(n) = I_{\{n\}}^{\{n\}}(n) = S_n(n)$ .

4. DEFINITION. For each pair  $m, n \in N^*$ ,  $S_m(n)$  and  $S^m(n)$  are called the simmetrical numbers relative to the diagonal of  $\mathcal{M}$ .

$S_m$  and  $S^m$  are called the simmetrical functions relative to the  $I$ -diagonal function  $I_{N^*}^{N^*}$ . As a rule,  $I_A^B$  and  $I_B^A$  are called the simmetrical functions relative to the  $I$ -diagonal function  $I_{N^*}^{N^*}$ .

5. DEFINITION. Let us consider the following rule  $\top: I \times I \rightarrow I$ ,  $I_A^B \top I_C^D = I_{A \cup C}^{B \cup D}$ . It is easy to see that  $\top$  is idempotent, commutative and associative, so that:

$$i) I_A^B \top I_A^B = I_A^B$$

$$ii) I_A^B \top I_C^D = I_C^D \top I_A^B$$

$$iii) (I_A^B \top I_C^D) \top I_E^F = I_A^B \top (I_C^D \top I_E^F), \text{ where } A, B, C, D, E, F \in \mathcal{P}(N^*) \setminus \emptyset$$

6. DEFINITION. Let us consider the following relative partial order relation  $\rho$ , where:

$$\rho \subset I \times I, \\ I_A^B \rho I_C^D \Leftrightarrow A \subset C \text{ and } B \subset D.$$

It is easy to see that  $(I, \top, \rho)$  is a semilattice.

7. **DEFINITION.** The elements  $u, v \in I$  are  $\rho$  - preceded if there is  $w \in I$  so that:

$$w \rho u \text{ and } w \rho v.$$

8. **DEFINITION.** The elements  $u, v \in I$ , are  $\rho$  - strictly preceded by  $w$  if:

i)  $w \rho u$  and  $w \rho v$ .

ii)  $\forall x \in I \setminus \{w\}$  so that  $x \rho u$  and  $x \rho v \Rightarrow x \rho w$ .

9. **DEFINITION.** Let us defined:

$$I^* = \{(u, v) \in I \times I \mid u, v \text{ are } \rho\text{-preceded}\}$$

$$I^\# = \{(u, v) \in I \times I \mid u, v \text{ are } \rho\text{-strictly preceded}\}.$$

It is evidently that  $(u, v) \in I^* \Leftrightarrow (v, u) \in I^*$  and  $(u, v) \in I^\# \Leftrightarrow (v, u) \in I^\#$ .

10. **DEFINITION.** Let us consider  $\top^\# = U \times U$ ,  $U \subset I$  and let us consider the following rule:

$\perp: I^\# \rightarrow W$ ,  $W \subset I$ ,  $I_A^B \perp I_C^D = I_{A \wedge C}^{B \wedge D}$  and the ordering partial relation  $r \subset U \times U$  so that  $I_A^B r I_C^D \Leftrightarrow I_C^D \rho I_A^B$ .

The structure  $(I^\#, \perp, r)$  is called the return of semilattice  $(I, \top, \rho)$ .

11. **DEFINITION.** The following set

$$\mathcal{B} = \{I_A^B \in I \mid A \cap B \neq \emptyset\}$$

is called the base of return  $(I^\#, \perp, r)$ .

12. **REMARK.** The base of return has the following properties:

i) if  $I_A^B \in \mathcal{B} \Rightarrow I_B^A \in \mathcal{B}$

ii) for  $\emptyset \neq X \subset N^*$ ,  $I_X^X \in \mathcal{B}$

iii) for  $I_A^B \in \mathcal{B}$  is true the following equivalence  $\emptyset \neq X \subset C_{N^*}(A \wedge B) \Leftrightarrow$  non existence of  $I_X^X \perp I_A^B$ .

13. **PROPOSITION.** For  $I_A^B \in \mathcal{B}$  there exists  $n \in N^*$  so that  $I_A^B(n) = I_{N^*}^{N^*}(n)$ .

*Proof.* Because  $A \cap B \neq \emptyset$  it results that there exists  $n \in A \wedge B$  so that:

$$I_A^B(n) = S_n(n) = I_{N^*}^{N^*}(n).$$

It results that for  $I_A^B \in \mathcal{B}$  then  $I_A^B$  has at least a point of contact with I-diagonal function.



14. **REMARK.** From the 1. it results:

$$I_{\{n\}}^B(n) = S_n(b_n), \text{ where } b_n = \begin{cases} b, & n < b = \min B \\ b_k, & b \leq n \leq b^* = \max B \\ \text{where} \\ b_k = \max\{x \in B \mid x \leq n\} \\ b^*, & n > b^* \end{cases}$$

and

$$I_A^{\{m\}}(m) = S^m(a_m), \text{ where } a_m = \begin{cases} a, & m < a = \min A \\ a_k, & a \leq m \leq a^* = \max A \\ \text{where} \\ a_k = \max\{x \in A \mid x \leq m\} \\ a^*, & m > a^* \end{cases}$$

15. **PROPOSITION.** *There are true the following equivalences:*

$$(I_A^B, I_C^D) \in I^\# \Leftrightarrow I_A^C, I_B^D \in \mathcal{B} \Leftrightarrow \exists n, m \in N^* \text{ so that:}$$

$$I_A^B(n) = I_{\{n\}}^B(n) = S_n(b_n), \quad I_C^D(n) = I_{\{n\}}^D(n) = S_n(d_n), \quad I_A^B(m) = I_A^{\{m\}}(m) = S^m(a_m), \text{ and} \\ I_C^D(m) = I_C^{\{m\}}(m) = S^m(c_m) \text{ where } a_m, b_n, c_m, d_n \text{ are defined in the sense of 14.}$$

If  $n \leq m$ , then  $n \leq a_m, c_m \leq m$ .

*Proof.* Evidently,

$$(I_A^B, I_C^D) \in I^\# \Leftrightarrow A \cap C \neq \emptyset \text{ and } B \cap D \neq \emptyset \Leftrightarrow I_A^C, I_B^D \in \mathcal{B}.$$

Because  $A \cap C \neq \emptyset$  and  $B \cap D \neq \emptyset$  it exists  $n \in A \cap C$  and  $m \in B \cap D$ . Then:

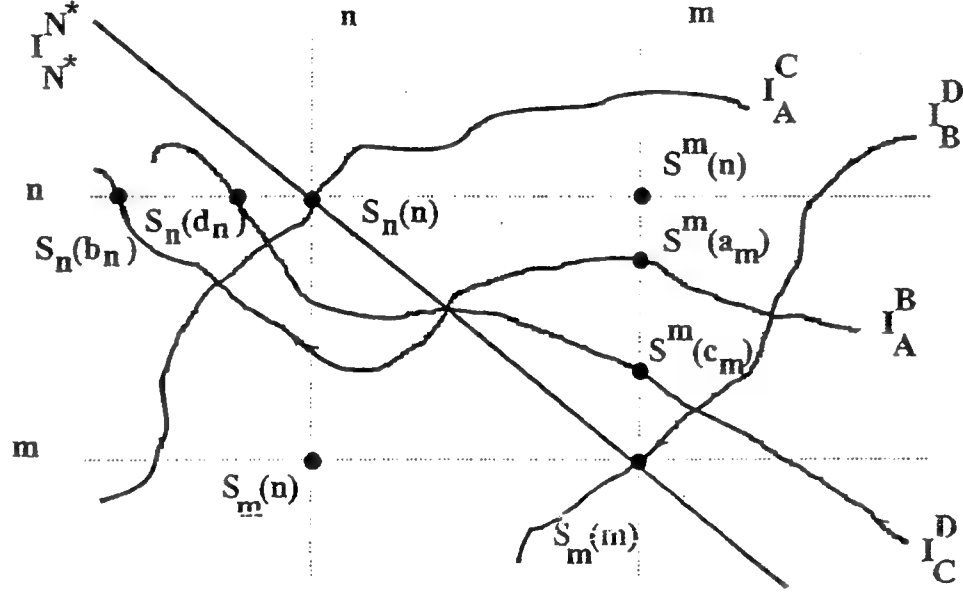
$$I_A^B(n) = I_{\{n\}}^B(n) = S_n(b_n), \quad I_C^D(n) = I_{\{n\}}^D(n) = S_n(d_n) \\ I_A^B(m) = I_A^{\{m\}}(m) = S^m(a_m), \quad I_C^D(m) = I_C^{\{m\}}(m) = S^m(c_m).$$

Conversely, if there exist  $n \in N^*$  so that  $I_A^B(n) = S_n(b_n)$  and  $I_C^D(n) = S_n(d_n)$ , then because  $I_A^B(n) = S_n(b_n)$  it results  $n = a_k = \max_i \{a_i \in A \mid a_i \leq n\}$ , so that  $n \in A$ . Because  $I_C^D(n) = S_n(d_n)$  it results  $n \in C$ .

Therefore  $A \cap C \neq \emptyset$ , thus, finally,  $I_A^C \in \mathcal{B}$ . It is also proved  $I_B^D \in \mathcal{B}$  in the same way.

If  $n \leq m$ , because  $n \in A \cap C$  it results that  $n \in \{x \in A \mid x \leq m\}$  and  $n \in \{y \in C \mid y \leq m\}$ , therefore  $n \leq a_m \leq m$  and  $n \leq c_m \leq m$ .

This is presented in the following scheme:



16. DEFINITION. The return  $(L^\#, \perp, r)$  of semilattice  $(L, \top, \rho)$  is:

- |                   |   |
|-------------------|---|
| a) null, if       | $L^\# = \{(u, u)   u \in L\} = \Delta_L$ .                  |
| b) weak, if       | $\text{card} L^\# < \text{card}(L \times L \setminus L^\#)$ |
| c) consistent, if | $\text{card} L^\# = \text{card}(L \times L \setminus L^\#)$ |
| d) vigour, if     | $\text{card} L^\# > \text{card}(L \times L \setminus L^\#)$ |
| e) total, if      | $L^\# = L \times L$ .                                       |

17. PROPOSITION. The return  $(I^\#, \perp, r)$  of the semilattice  $(I, \top, \rho)$  is consistent.

*Proof.* Evidently,  $\text{card}(\mathcal{P}(N^*) \setminus \emptyset) = \aleph$ ,  $\text{card } I = \text{card}[(\mathcal{P}(N^*) - \emptyset) \times (\mathcal{P}(N^*) - \emptyset)] = \aleph$  and  $\text{card}(I \times I) = \aleph$ .

Let us consider  $\mathcal{F} = \{(A, C) | A, C \in \mathcal{P}(N^*) - \emptyset, A \cap C = \emptyset\}$  and  $\overline{\mathcal{F}} = \{(A, C) | A, C \in \mathcal{P}(N^*) - \emptyset, A \cap C \neq \emptyset\}$ .

$\text{card } \mathcal{F} = \text{card } \overline{\mathcal{F}} = \aleph$ . Indeed, if  $A \cap C = \emptyset$  it results that  $C_{N^*} \cdot A \cup C_{N^*} \cdot C = N^*$ ; because for every  $X \in \mathcal{P}(N^*) - \emptyset \exists Y = N^* \setminus X$  so that  $X \cup Y = N^*$  then it results  $\text{card } \mathcal{F} = \text{card } \mathcal{P}(N^*) = \aleph$ . Because for each  $(A, C)$ ,  $A, C \in \mathcal{P}(N^*) - \emptyset, A \cap C = \emptyset$ , it exist at least two  $(A_1, C_1), (A_2, C_2)$  with  $A_1 \cap C_1 \neq \emptyset, A_2 \cap C_2 \neq \emptyset$  it results  $\text{card } \overline{\mathcal{F}} \geq \text{card } \mathcal{F} = \aleph$ .

Since  $\text{card} \overline{\mathcal{F}} \leq \text{card}[(\mathcal{P}(N^*) - \emptyset) \times (\mathcal{P}(N^*) - \emptyset)] = \aleph$  finally  $\text{card} \overline{\mathcal{F}} = \aleph$ . Because  $\text{card} I^\# = \text{card}(\overline{\mathcal{F}} \times \overline{\mathcal{F}}) = \aleph$  and  $\text{card}(I \times I) - I^\# = \text{card}(\mathcal{F} \times \mathcal{F}) = \aleph$  it results that  $(I^\#, \perp, r)$  is a return consistent.

18. **REMARK.** Generaly, it is interesting the folowing problems:

i) what relations, operations, structures can be defined on

$$M = \{S_m(n) \mid n, m \in N^*\}?$$

ii) what relations, operations, structures can be defined on

$$\mathcal{H} = \{f \mid f: N^* \rightarrow \mathcal{M}\}?$$

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# ON A FUNCTION IN THE NUMBERS THEORY

by

Ion Cojocaru and Sorin Cojocaru

**Abstract.** In the present paper we study some series concerning the following function of the Numbers Theory [1]: " $S : \mathbb{N} \rightarrow \mathbb{N}$  such that  $S(n)$  is the smallest  $k$  with property that  $k!$  is divisible by  $n$ ".

1. **Introduction.** The following functions in Numbers Theory are well - known : the function  $\mu(n)$  of M $\ddot{o}$ bius, the function  $\xi(s)$  of Riemann ( $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, s = \sigma + it \in \mathbb{C}$ ), the function  $\Lambda(n)$  of Mangoldt  $\left( \Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \\ 0, & \text{if } n \neq p^m \end{cases} \right)$  etc.

The purpose of this paper is to study some series concerning the following function of the Numbers Theory "[1]  $S : \mathbb{N} \rightarrow \mathbb{N}$  such that  $S(n)$  is the smallest integer  $k$  with the propriety that  $k!$  is divisible by  $n$ ".

We first prove the divergence of some series involving the  $S$  function, using an unitary method, and then we prove that the series  $\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)}$  is convergent to a number  $S \in (71/100, 101/100)$  and we study some applications of this series in the Numbers Theory .

Then we prove that series  $\sum_{n=2}^{\infty} \frac{1}{S(n)!}$  is convergent to a real numbers  $s \in (0.717, 1.253)$  and that the sum of the remarkable series  $\sum_{n \geq 2} \frac{S(n)}{n!}$  is a irrational number.

## 2. The main results

**Proposition 1.** If  $(x_n)_{n \geq 1}$  is strict increasing sequence of natural numbers, then the series :

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}, \quad (1)$$

is divergent.

**Proof.** We consider the function  $f: [x_n, x_{n+1}] \rightarrow \mathbb{R}$ , defined by  $f(x) = \ln \ln x$  is meets the conditions of the Lagrange's theorem of finite increases. Therefore there is  $c_n \in (x_n, x_{n+1})$  such that :

$$\ln \ln x_{n+1} - \ln \ln x_n = \frac{1}{c_n \ln c_n} (x_{n+1} - x_n). \quad (2)$$

Because  $x_n < c_n < x_{n+1}$ , we have :

$$\frac{x_{n+1} - x_n}{x_{n+1} \ln x_{n+1}} < \ln \ln x_{n+1} - \ln \ln x_n < \frac{x_{n+1} - x_n}{x_n \ln x_n}, (\forall) n \in \mathbb{N}, \quad (3)$$

if  $x_n \neq 1$ .

We know that for each  $n \in \mathbb{N}^* \setminus \{1\}$ ,  $\frac{S(n)}{n} \leq 1$ , i.e.

$$0 < \frac{S(n)}{n \ln n} \leq \frac{1}{\ln n}, \quad (4)$$

from where it results that  $\lim_{n \rightarrow \infty} \frac{S(n)}{n \ln n} = 0$ . Hence there is  $k > 0$  such that  $\frac{S(n)}{n \ln n} < k$ , i.e.,  $n \ln n > \frac{S(n)}{k}$  for any  $n \in \mathbb{N}^*$ , so

$$\frac{1}{x_n \ln x_n} < \frac{k}{S(x_n)}. \quad (5)$$

Introducing (5) in (3) we obtain :

$$\ln \ln x_{n+1} - \ln \ln x_n < k \frac{x_{n+1} - x_n}{S(x_n)}, (\forall) n \in \mathbb{N}^* \setminus \{1\}. \quad (6)$$

Summing up after  $n$  it results :

$$\sum_{n=1}^m \frac{x_{n+1} - x_n}{S(x_n)} > \frac{1}{k} (\ln \ln x_{m+1} - \ln \ln x_1).$$

Because  $\lim_{m \rightarrow \infty} x_m = \infty$  we have  $\lim_{m \rightarrow \infty} \ln \ln x_m = \infty$ , i.e., the series :

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$$

is divergent. The Proposition 1 is proved.

**Proposition 2.** Series  $\sum_{n=2}^{\infty} \frac{1}{S(n)}$  is divergent.

**Proof.** We use Proposition 1 for  $x_n = n$ .

**Remarks.**

- 1) If  $x_n$  is the  $n$ -th prime number, then the series  $\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$  is divergent.
- 2) If the sequence  $(x_n)_{n \geq 1}$  forms an arithmetical progression of natural numbers, then the series  $\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$  is divergent.
- 3) The series  $\sum_{n=1}^{\infty} \frac{1}{S(2n+1)}$ ,  $\sum_{n=1}^{\infty} \frac{1}{S(4n+1)}$  etc., are all divergent.

In conclusion, Proposition 1 offers us an unitary method to prove that the series having one of the precedent forms are divergent.

**Proposition 3.** The series

$$\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)}$$

is convergent to a number  $s \in (71/100, 101/100)$ .

**Proof.** From the definition it results  $S(n) \leq n!$ ,  $(\forall) n \in \mathbb{N}^* \setminus \{1\}$ , so  $\frac{1}{S(n)} \geq \frac{1}{n!}$ .

Summing up, beginning with  $n=2$  we obtain :

$$\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2.$$

The product  $S(2)S(3)\dots S(n)$  is greater than the product of prime numbers from the set  $\{1, 2, \dots, n\}$ , because  $S(p)=p$ , for  $p$ = prime number. Therefore :

$$\frac{1}{\prod_{i=2}^n S(i)} < \frac{1}{\prod_{i=2}^n p_i}, \quad (7)$$

where  $p_k$  is the biggest number smaller or equal to  $n$ .

There are the inequalities :

$$S = \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)} = \frac{1}{S(2)} + \frac{1}{S(2)S(3)} + \frac{1}{S(2)S(3)S(4)} + \dots + \frac{1}{S(2)S(3)\dots S(k)} + \dots < \frac{1}{2} + \frac{2}{2 \cdot 3} + \frac{2}{2 \cdot 3 \cdot 5} + \frac{2}{2 \cdot 3 \cdot 5 \cdot 7} + \frac{2}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11} + \dots + \frac{p_{k+1} - p_k}{p_1 p_2 \dots p_k} + \dots \quad (8)$$

Using the inequality  $p_1 p_2 \dots p_k > p_{k+1}^3$ ,  $(\forall) k \geq 5$  [5], we obtain :

$$S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{1}{p_6^2} + \frac{1}{p_7^2} + \dots + \frac{1}{p_{k+1}^2} + \dots \quad (9)$$

We symbolise by  $P = \frac{1}{p_6^2} + \frac{1}{p_7^2} + \dots$  and observe that  $P < \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + \dots$

It results :

$$P < \frac{\pi^2}{6} - \left( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{12^2} \right),$$

where

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \text{ (EULER).}$$

Introducing in (9) we obtain :

$$S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{1}{105} + \frac{\pi^2}{6} - 1 - \frac{1}{2^2} - \frac{1}{3^2} - \dots - \frac{1}{12^2}.$$

Estimating with an approximation of an order not more than  $\frac{1}{10^2}$ , we find :

$$0,71 < \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)} < 0,79. \quad (10)$$

The proposition 3 is proved.

**Remark.** Giving up at the right increase from the first terms in the inequality (8) we can obtain a better right ranging :

$$\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)} < 0,97. \quad (11)$$

**Proposition 4.** Let  $\alpha$  be a fixed real number,  $\alpha \geq 1$ . Then the series  $\sum_{n=2}^{\infty} \frac{n^\alpha}{S(2)S(3)\dots S(n)}$

is convergent.

**Proof.** Be  $(x_k)_{k \geq 1}$  the sequence of prime numbers. We can write :

$$\frac{2^\alpha}{S(2)} = \frac{2^\alpha}{2} = 2^{\alpha-1}$$

$$\frac{3^\alpha}{S(2)S(3)} = \frac{3^\alpha}{p_1 p_2}$$

$$\frac{4^\alpha}{S(2)S(3)S(4)} < \frac{4^\alpha}{p_1 p_2} < \frac{p_3^\alpha}{p_1 p_2}$$

$$\frac{5^\alpha}{S(2)S(3)S(4)S(5)} < \frac{5^\alpha}{p_1 p_2 p_3} < \frac{p_4^\alpha}{p_1 p_2 p_3}$$

$$\frac{6^\alpha}{S(2)S(3)S(4)S(5)S(6)} < \frac{6^\alpha}{p_1 p_2 p_3} < \frac{p_4^\alpha}{p_1 p_2 p_3}$$

$$\frac{n^\alpha}{S(2)S(3)\dots S(n)} < \frac{n^\alpha}{p_1 p_2 \dots p_k} < \frac{p_{k+1}^\alpha}{p_1 p_2 \dots p_k},$$

where  $p_i \leq n, i \in \{1, \dots, k\}, p_{k+1} > n$ .

Therefore

$$\sum_{n=2}^{\infty} \frac{n^{\alpha}}{S(2)S(3)\dots S(n)} < 2^{\alpha-1} + \sum_{n=2}^{\infty} \frac{(p_{k+1} - p_k) \cdot p_{k+1}^{\alpha}}{S(2)S(3)\dots S(n)} < \\ < 2^{\alpha-1} + \sum_{n=2}^{\infty} \frac{p_{k+1}^{\alpha}}{p_1 p_2 \dots p_k}.$$

Then it exists  $k_0 \in \mathbb{N}$  such that for any  $k \geq k_0$  we have :

$$p_1 p_2 \dots p_k > p_{k+1}^{\alpha+3}.$$

Therefore

$$\sum_{n=2}^{\infty} \frac{n^{\alpha}}{S(2)S(3)\dots S(n)} < 2^{\alpha-1} + \sum_{k=1}^{k_0-1} \frac{p_{k+1}^{\alpha+1}}{p_1 p_2 \dots p_k} + \sum_{k \geq k_0} \frac{1}{p_{k+1}^2}.$$

Because the series  $\sum_{k \geq k_0} \frac{1}{p_{k+1}^2}$  is convergent it results that the given series is convergent too.

**Consequence 1.** It exists  $n_0 \in \mathbb{N}$  so that for each  $n \geq n_0$  we have  $S(2)S(3)\dots S(n) > n^{\alpha}$ .

**Proof.** Because  $\lim_{n \rightarrow \infty} \frac{n^{\alpha}}{S(2)S(3)\dots S(n)} = 0$ , there is  $n_0 \in \mathbb{N}$  so that  $\frac{n^{\alpha}}{S(2)S(3)\dots S(n)} < 1$

for each  $n \geq n_0$ .

**Consequence 2.** It exists  $n_0 \in \mathbb{N}$  so that :

$$S(2) + S(3) + \dots + S(n) > (n-1)n^{\frac{\alpha}{n-1}} \text{ for each } n \geq n_0.$$

**Proof.** We apply the inequality of averages to the numbers  $S(2), S(3), \dots, S(n)$  :

$$S(2) + S(3) + \dots + S(n) > (n-1)^{n-1} \sqrt[n-1]{S(2)S(3)\dots S(n)} > (n-1)n^{\frac{\alpha}{n-1}}, \forall n \geq n_0.$$

We can write it as it follows :

$$\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{3!} + \dots = \frac{1}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{8}{5!} + \frac{14}{6!} + \dots = \sum_{n=2}^{\infty} \frac{a(n)}{n!}, \text{ where } a(n) \text{ is the}$$

number of solutions for the equation  $S(x) = n, n \in \mathbb{N}, n \geq 2$ .

It results from the equality  $S(x)=n$  that  $x$  is a divisor of  $n!$ , so  $a(n)$  is smaller than  $d(n!)$ .

So,  $a(n) < d(n!)$ .

**Lemma 1.** We have the inequality :

$$d(n) \leq n-2, \text{ for each } n \in \mathbb{N}, n \geq 7. \quad (12)$$

**Proof.** Be  $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$  with  $p_1, p_2, \dots, p_k$  prime numbers, and  $a_i \geq 1$  for each  $i \in \{1, 2, \dots, k\}$ . We consider the function  $f: [1, \infty) \rightarrow \mathbb{R}, f(x) = a^x - x - 2, a \geq 2$ , fixed. It is derivable on  $[1, \infty)$  and  $f'(x) = a^x \ln a - 1$ . Because  $a \geq 2$ , and  $x \geq 1$  it results that  $a^x \geq 2$ , so  $a^x \ln a \geq 2 \ln a = \ln a^2 \geq \ln 4 > \ln e = 1$ ,  $f'(x) > 0$  for each  $x \in [1, \infty)$  and  $a \geq 2$ , fixed. But  $f(1) = a-3$ . It results that for  $a \geq 3$  we have  $f(x) \geq 0$  means  $a^x \geq x+2$ .

Particularly, for  $a = p_i, i \in \{1, 2, \dots, k\}$ , we obtain  $p_i^{a_i} \geq a_i + 2$  for each  $p_i \geq 3$ .

If  $n = 2^s, s \in \mathbb{N}^*,$  then  $d(n) = s+1 < 2^s-2 = n-2$  for  $s \geq 3$ .

So we can assume  $k \geq 2$ , i.e.  $p_2 \geq 3$ . The following inequalities result :

$$p_1^{a_1} \geq a_1 + 1$$

$$p_2^{a_2} \geq a_2 + 1$$

$$\dots$$

$$p_k^{a_k} \geq a_k + 1,$$

equivalent with

$$p_1^{a_1} \geq a_1 + 1, p_2^{a_2} - 1 \geq a_2 + 1, \dots, p_k^{a_k} - 1 \geq a_k + 1. \quad (13)$$

By multiplying, member with member, of the inequalities (13) we obtain :

$$p_1^{a_1}(p_2^{a_2} - 1) \dots (p_k^{a_k} - 1) \geq (a_1 + 1)(a_2 + 1) \dots (a_k + 1) = d(n). \quad (14)$$

Considering the obvious inequality :

$$n - 2 \geq p_1^{a_1}(p_2^{a_2} - 1) \dots (p_k^{a_k} - 1) \quad (15)$$

and using (14) it results that :

$$n - 2 \geq d(n) \text{ for each } n \geq 7.$$

**Lemma 2.**  $d(n!) < (n-2)!$  for each  $n \in \mathbb{N}$ ,  $n \geq 7$ . (16)

**Proof.** We carry out an induction after  $n$ . So, for  $n=7$ ,

$$d(7!) = d(2^4 \cdot 2^3 \cdot 5 \cdot 7) = 60 < 120 = 5!.$$

We assume that  $d(n!) < (n-2)!$ .

$$d((n+1)!) = d(n!(n+1)) < d(n!) d(n+1) < (n-2)! d(n+1) < (n-2)!(n-1) = (n-1)!,$$

because in according to Lemma 1,  $d(n+1) < n-1$ .

**Proposition 5.** The series  $\sum_{n=2}^{\infty} \frac{1}{S(n)!}$  is convergent to a number  $s \in (0.717, 1.253)$ .

**Proof.** From Lemma 2 it results that  $a(n) < (n-2)!$ , so  $\frac{a(n)}{n!} < \frac{1}{n(n-1)}$  for every  $n \in \mathbb{N}$ ,

$$n \geq 7 \text{ and } \sum_{n=2}^{\infty} \frac{1}{S(n)!} = \sum_{n=2}^6 \frac{a(n)}{n!} + \sum_{n=7}^{\infty} \frac{1}{(n-1)!}.$$

$$\text{Therefore } \sum_{n=2}^{\infty} \frac{a(n)}{n!} < \frac{1}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{8}{5!} + \frac{14}{6!} + \sum_{n=7}^{\infty} \frac{1}{n^2 - n}. \quad (17)$$

$$\text{Because } \sum_{n=2}^{\infty} \frac{1}{n^2 - n} = 1 \text{ we have there is a number } s > 0, s = \sum_{n=2}^{\infty} \frac{1}{S(n)!}.$$

From (17) we obtain :

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{S(n)!} &< \frac{391}{360} + 1 - \frac{1}{2^2 - 2} - \frac{1}{3^2 - 3} - \frac{1}{4^2 - 4} + \\ &+ \frac{1}{5^2 - 5} + \frac{1}{6^2 - 6} = \frac{751}{360} - \frac{5}{6} = \frac{451}{360} < 1,253. \end{aligned}$$

But, because  $S(n) \leq n$  for every  $n \in \mathbb{N}^*$ , it results :

$$\sum_{n=2}^{\infty} \frac{1}{S(n)!} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2.$$

Consequently, for the number  $s$  we obtain the range  $e-2 < s < 1,253$ , i.e.,  $0,717 < s < 1,253$ .

Because  $S(n) \leq n$ , it results  $\sum_{n \geq 2} \frac{S(n)}{n!} \leq \sum_{n \geq 2} \frac{1}{(n-1)!}$ . Therefore the series  $\sum_{n \geq 2} \frac{S(n)}{n!}$  is convergent to a number  $f$ .

**Proposition 6.** The sum  $f$  of the series  $\sum_{n \geq 2} \frac{S(n)}{n!}$  is an irrational number.

**Proof.** From the above results that  $\lim_{n \rightarrow \infty} \sum_{i=2}^n \frac{S(i)}{i!} = 1$ . Under these circumstances that  $f \in \mathbb{Q}$ ,  $f > 0$ . Therefore it exists  $a, b \in \mathbb{N}$ ,  $(a, b) = 1$ , so that  $f = \frac{a}{b}$ .

Let  $p$  be a fixed prime number,  $p > b$ ,  $p \geq 3$ . Obviously,  $\frac{a}{b} = \sum_{i=2}^{p-1} \frac{S(i)}{i!} + \sum_{i \geq p} \frac{S(i)}{i!}$  which leads to :



$$\frac{(p-1)!a}{b} = \sum_{i=2}^{p-1} \frac{(p-1)!S(i)}{i!} + \sum_{i \geq p} \frac{(p-1)!S(i)}{i!}.$$

Because  $p > b$  results that  $\frac{(p-1)!a}{b} \in \mathbb{N}$  and  $\sum_{i=2}^{p-1} \frac{(p-1)!S(i)}{i!} \in \mathbb{N}$ . Consequently we have  $\sum_{i \geq p} \frac{(p-1)!S(i)}{i!} \in \mathbb{N}$  too.

Be  $\alpha = \sum_{i \geq p} \frac{(p-1)!S(i)}{i!} \in \mathbb{N}$ . So we have the relation

$$\alpha = \frac{(p-1)!S(p)}{p!} + \frac{(p-1)!S(p+1)}{(p+1)!} + \frac{(p-1)!S(p+2)}{(p+2)!} + \dots$$

Because  $p$  is a prime number it results  $S(p)=p$ .

So

$$\alpha \approx 1 + \frac{S(p+1)}{p(p+1)} + \frac{S(p+2)}{p(p+1)(p+2)!} + \dots > 1 \quad (18)$$

We know that  $S(p+1) \leq p+1 (\forall i \geq 1)$ , with equality only if the number  $p+i$  is prime.

Consequently, we have

$$\alpha < 1 + \frac{1}{p} + \frac{1}{p(p+1)} + \frac{1}{p(p+1)(p+2)} + \dots < 1 + 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \dots = \frac{p}{p-1} < 2 \quad (19)$$

From the inequalities (18) and (19) results that  $1 < \alpha < 2$ , impossible, because  $\alpha \in \mathbb{N}$ .

The proposition is proved.

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# On Smarandache's form of the individual Fermat–Euler theorem

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*Dedicated to the memory of the late Professor Štefan Schwarz*

## Abstract

In the paper it is shown how a form of the classical FERMAT–EULER Theorem discovered by F.SMARANDACHE fits into the generalizations found by Š.SCHWARZ, M.LAŠŠÁK and the author. Then we show how SMARANDACHE's algorithm can be used to effective computations of the so called group membership.

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# 1 Variations on Fermat–Euler theorem

In [Schw81] a semigroup approach to the FERMAT–EULER Theorem was developed

$$a^{\varphi(n)} \equiv 1 \pmod{n}, \quad (a, n) = 1$$

based on an idempotent technique giving the best possible extensions of this fundamental result to the set  $Z$  of the all integers. In [LaPo96] the idea was generalized to finite commutative rings  $R$  and subsequently to the *residually finite* DEDEKIND domains, that is DEDEKIND domains  $R$  satisfying the finiteness condition:

(FN) *For every non-zero ideal  $M \subset R$  the residue class ring  $R/M$  is finite.*

A detailed specialization of these results depends then upon a corresponding detailed knowledge of the structure of the group of units (i.e. invertible elements) of the corresponding residue class ring  $R/M$ . The most known prototypes of rings where this knowledge is available are, besides  $Z_n$  the ring of residue classes modulo  $n$ , the algebraic number fields. Thus for instance, for residually finite DEDEKIND domains we only have Lemma 8 in general. For algebraic number fields see [LaPo96].

## 1.1 Semigroup level

The basic underlying idea of the proofs of generalizations of FERMAT–EULER Theorem given in [Schw81] and [LaPo96] is based on the some elementary semigroup ideas. To describe them we shall suppose in this Section that  $S$  is a finite commutative semigroup written multiplicatively.

Given an  $x \in S$ , the sequence

$$x, x^2, x^3, \dots, x \in S \tag{1}$$

contains some of its elements multiple times. If we denote by  $k = k(x) \in N$  (here  $N$  is the set of positive integers) the least such exponent for which  $x^k$  appears at least twice in (1) and  $d = d(x)$  the least exponent with  $x^k = x^{k+d}$ , then the sequence (1) has the form

$$x, x^2, \dots, x^{k-1}, x^k, \dots, x^{k+d-1}, x^k, \dots$$

The next elementary result is instrumental in the investigations which follow:

**Lemma 1 (Frobenius 1895)** *For every  $x \in S$  the set*

$$C(x) = \{x^k, \dots, x^{k+d-1}\} \tag{2}$$

*forms a cyclic group with respect to the multiplication.*

The identity element  $\mathbf{1} = x^r, r = r(x)$  of the group  $C(x)$  is the unique idempotent of  $R$  which belongs to (1). This connections are described saying *the element  $x$  belongs to the idempotent  $e$* .

The above observations imply (see also proof of Theorem 1.9 in [LaPo96]):

**Proposition 1 (Individual Fermat – Euler Theorem)** *If  $\kappa, \delta \in N$  with  $\kappa \geq k(x), d(x) \mid \delta$ , then for every  $x \in S$  we have*

$$x^{\kappa+\delta} = x^{\kappa}$$

*and the numbers  $k(x)$  and  $d(x)$  are the least positive numbers possessing this property.*

The main problem here is to determine the exact values of  $k(x)$  and  $d(x)$ . As mentioned above, more knowledge about  $S$  is required for this task. In the process of the determination of values of these numbers further structural results are needed. To make the paper self-contained we shall outline some crucial facts, the reader is referred to [LaPo96] for more details. Of basic importance are properties of the idempotents.

Let  $E_S$  denote the set of idempotents of  $S$ . Let  $e \in E_S$ . Then the set

$$P^S(e) = \{x \in S ; x \text{ belongs to } e\}$$

is the largest subsemigroup of  $S$ , which except for  $e$  contains no other idempotent of  $S$ . This uniquely determined maximal subsemigroup  $P^S(e)$  will be called the *maximal (multiplicative) semigroup (of semigroup  $S$ ) belonging to the idempotent  $e \in E_S$* . Note that

$$S = \bigcup_{e \in E_S} P^S(e).$$

Moreover, if  $e \in E_S$  is an idempotent in  $S$ , then there always exists a subgroup of  $S$  containing  $\mathbf{1}$  as its identity, e.g. the group  $\{e\}$  or the group  $C(x)$  of Lemma 1 provided  $x$  belongs to the idempotent  $e$ . Since  $S$  is finite, there exist maximal subgroup of  $S$  amongst the all subgroups of  $S$  for which  $e$  serves as the identity element. We shall call this group  $G^S(e)$  the *maximal (multiplicative) subgroup of  $S$  belonging to the idempotent  $e \in E_S$* . It is surprising that the existence of these subgroups is almost unknown in the classical number theory.

Given an idempotent  $e \in E_S$ , define

$$k_e = \max\{k(x) ; x \in P^S(e)\}, \quad d_e = \text{l.c.m.}\{d(x) ; x \in P^S(e)\}$$

and

$$k_S = \max\{k(x) ; x \in S\}, \quad d_S = \text{l.c.m.}\{d(x) ; x \in S\}.$$

The algebraic meaning of numbers  $k(x), d(x)$  for  $x \in S$ ,  $k_e, d_e$  for  $e \in E_S$ ,  $k_S$ , and  $d_S$  is best explained by the next results [LaPo96, p.268]:

**Lemma 2** For any  $x \in S$

- (a) Every of  $x^{k(x)}, x^{k_e}, x^{k_S}$  is an element of a subgroup of  $S$ . More precisely,  $x^{k(x)} \in C(x)$ ,  $x^{k_e} \in G^S(e)$  for  $x \in P_S(e)$ , and  $x^{k_S} \in \bigcup_{f \in E_S} G^S(f)$ .
- (b) For every  $x \in \bigcup_{f \in E_S} G^S(f)$  the element  $x^{d(x)} = x^{d_e} = x^{d_S}$  is an idempotent of  $S$ .

These numbers enable us to complement the above individual FERMAT-EULER Theorem and its classical version to statements over three basic sub-semigroup levels of  $S$ , namely:

- the least subsemigroup generated by  $x$  yielding FERMAT-EULER Theorems of individual type,
- the maximal subsemigroup belonging to an idempotent of  $S$  yielding local types of this Theorem, and
- the whole multiplicative semigroup of  $S$  giving global type FERMAT-EULER Theorems.

Namely, it follows from the definitions of numbers  $k_e, d_e, k_S, d_S$  and Theorems 1.10, and 1.11 of [LaPo96] that:

**Proposition 2 (Local Fermat – Euler theorem)** If  $e \in E_S$ , and  $\kappa, \delta \in N$  with  $\kappa \geq k_e, d_e \mid \delta$ , then then for every  $x \in P^S(e)$  we have

$$x^{\kappa+\delta} = x^\kappa.$$

Moreover, the numbers  $k_e, d_e$  are the least positive integers such that this equality holds under the given conditions for each  $x \in P^S(e)$ .

**Proposition 3 (Global Fermat – Euler theorem)** For every  $x \in S$  and  $\kappa, \delta \in N$  with  $\kappa \geq k_S, d_S \mid \delta$  we have

$$x^{\kappa+\delta} = x^\kappa$$

and the numbers  $k_S, d_S$  are the least positive integers such that this equality holds under the given conditions for each  $x \in S$ .

## 1.2 Finite rings level

The classical FERMAT-EULER Theorem involves both additive and multiplicative structure of the ring of integers, so it seems unavoidable to respect the interference of both, the additive and multiplicative structure of the underlying ring in the process to find the best possible generalization of this Theorem joining its classical form.

Therefore, in this section we shall always suppose that  $R$  denotes a finite commutative ring with the identity element  $1 = 1_R$ . The set  $E_R$  of idempotents

of  $R$  is obviously non empty for  $0, 1 \in E$  and it is finite. The set  $E_R$  can be endowed with a partial ordering

$$x \leq y \iff xy = x.$$

An idempotent  $e \in E_R$  is called *primitive* if it is minimal in the ordered set  $(E_R \setminus \{0\}, \leq)$ .

**Lemma 3** Let  $e_1, \dots, e_n$  be the all primitive idempotents of  $R$ . Then

(i) If  $0 \neq f \in E$ , then

$$fe_i = \begin{cases} e_i & \text{if } e_i \leq f, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If  $0 \neq f \in E$ , then

$$f = \sum_{\substack{i=1 \\ fe_i = e_i}}^n e_i.$$

To simplify the notation, given  $f \in E_R$ , denote

$$\begin{aligned} I_f &= \{i \in \{1, \dots, n\} ; fe_i = e_i\}, \\ I'_f &= \{1, \dots, n\} \setminus I_f. \end{aligned}$$

Note the following facts (the reader is referred for more details to [LaPo96]) for  $e \in E_R$ :

- $G^R(e) = P^R(e)e$ , thus in particular  $G^R(1) = P^R(1)$ ,
- $G^R(e)$  is the group of units of  $eR$  with respect to the ring multiplication and  $G^R(e) = P^{eR}(e)$ .
- $P^R(0) = N(R)$ , where  $N(R)$  denotes the the *nil-radical* of the ring  $R$

$$N(R) = \{x \in R ; x^t = 0 \text{ for some } t > 0\}$$

which is formed by nilpotent elements of  $R$ . Thus nil-radical is the maximal semigroup belonging to the idempotent 0.

If  $e_1, \dots, e_n$  are all the primitive idempotents of  $R$ , then we have the *Peirce decomposition* of  $R$

$$R = e_1 R \oplus \dots \oplus e_n R,$$

and ([LaPo96, p.263–264])

$$\begin{aligned} P^R(f) &= P^{e_1 R}(e_1 f) \oplus \dots \oplus P^{e_n R}(e_n f) = \bigoplus_{i \in I_f} G^R(e_i) \oplus \bigoplus_{i \in I'_f} N(e_i R) \\ &= G^R(f) \oplus N((1-f)R) \\ G^R(f) &= \bigoplus_{i \in I_f} G^R(e_i). \end{aligned} \tag{3}$$

Important observation is given in the next result:

**Lemma 4** ([LaPo96, Theorem 1.14]) *Let  $e_1, \dots, e_n \in E$  be the primitive idempotents of  $R$ . Then for every  $i = 1, \dots, n$  we have*

$$e_i R = G^R(e_i) \cup N(e_i R) \quad (4)$$

*and this union is disjoint.*

If we define for  $y \in e_i R$

$$\nu_i(y) = \begin{cases} 1 & \text{if } y \in G^R(e_i), \\ t & \text{if } y \in N(e_i R), \text{ where } t \text{ is minimal with } y^t = 0. \end{cases}$$

and

$$\nu(x) = \max\{\nu_i(e_i x) ; i = 1, \dots, n\},$$

then we have:

**Lemma 5** ([LaPo96, Corollary 1 of Theorem 1.15]) *For every  $x \in R$  we have  $k(x) = \nu(x)$ .*

Finally, if we define

$$\nu^{(i)} = \max\{\nu(x) ; x \in e_i R\}, \quad \mu^{(i)} = \text{l.c.m.}\{d(x) ; x \in G^R(e_i)\}$$

for every  $i = 1, \dots, n$  and

$$\nu_f = \max\{\nu^{(i)} ; i \in I_f\}, \quad \mu_f = \text{l.c.m.}\{\mu^{(i)} ; i \in I_f\}$$

then numbers  $\mu_f, f \in E_R$ , have the following property ([LaPo96, Lemma 1.8, Corollary 1]):

**Lemma 6** *If  $f \in E_R$ , then  $\mu_f | \mu_1$  and the number  $\mu_f$  is the exponent of the group  $G^R(f)$ .*

If analogically we define

$$\nu_R = \max\{\nu_f ; f \in E\} = \nu_0, \quad \mu_R = \text{l.c.m.}\{\mu_f ; f \in E\} = \mu_1,$$

then these are the least positive integers such that:

**Lemma 7** (a)  $x^{\nu_R}$  is an element of a multiplicative subgroup of  $(R, \cdot)$  for every  $x \in R$ ,

(b)  $x^{\mu_R}$  is an idempotent for every  $x \in \bigcup_{f \in E} G^R(f)$ .

The previous considerations together give the following generalized FERMAT-EULER Theorems (global and local) which are "computationally easier" to handle in comparison with the Proposition 2 and 3, because it reduces the determination of the values of  $\nu^{(i)}, \mu^{(i)}$  for  $i = 1, \dots, n$  to the knowledge of the values  $\nu_f, \mu_f$  for every  $f \in E_R$ . Thus we have:

**Proposition 4 (Local Fermat – Euler theorem)** *If  $e \in E_R$ , and  $\kappa, \delta \in N$  with  $\kappa \geq \nu_e, \mu_e \mid \delta$ , then then for every  $x \in P^R(e)$  we have*

$$x^{\kappa+\delta} = x^\kappa.$$

*The numbers  $\nu_e, \mu_e$  are the least positive integers such that this equality holds under the given conditions for each  $x \in P^R(e)$ .*

**Proposition 5 (Global Fermat – Euler theorem)** *For every  $x \in R$  and  $\kappa, \delta \in N$  with  $\kappa \geq \nu_R, \mu_R \mid \delta$  we have*

$$x^{\kappa+\delta} = x^\kappa$$

*and the numbers  $\nu_R, \mu_R$  are the least positive integers such that this equality holds under the given conditions for each  $x \in R$ .*

### 1.3 Dedekind domains level

Henceforth we shall suppose that  $R$  stands for a residually finite DEDEKIND domain. If  $M$  is a non-zero ideal of  $R$  then the residue class ring  $R/M$  will be denoted by  $R_M$  and its elements by  $[x] = [x]_M = x + M$  for  $x \in R$ . The norm  $N(M)$  of an ideal  $M$  is defined as the cardinality of the residue class ring  $R_M$ .

Since every proper ideal  $M$  of a DEDEKIND domain  $R$  is uniquely (up to the order of the factors) expressible in the form of a product of powers of prime ideals, suppose that

$$M = P_1^{u_1} \dots P_r^{u_r}, \quad (5)$$

where  $P_1, \dots, P_r$  are distinct prime ideals of  $R$  and  $u_i > 0, i = 1, \dots, r$ .

For these rings the FERMAT-EULER Theorem is usually stated in the form:

**Lemma 8 ([Nark74, Theorem 1.8])** *Let  $G^{R_M}([1]_M)$  denote the group of units of the residue class ring  $R_M$  with  $M \neq (0)$  of a residually finite Dedekind domain  $R$ . If  $\varphi_R(M) = \text{card}(G^{R_M}([1]_M))$ , then*

$$\varphi_R(M) = N(M) \prod_{\mathcal{P}} (\infty - N(\mathcal{P})^{-\infty}),$$

*where the product is extended over all prime ideals appearing in (5), and, moreover, if  $x \in R$  and  $((x), M) = (1)$ , then*

$$x^{\varphi_R(M)} \equiv 1 \pmod{M}.$$

As usual, we say that an ideal  $A$  divides an ideal  $B$ , in symbols  $A \mid B$ , if there exists an ideal  $C$  with  $B = AC$ . It can be easily shown that in a DEDEKIND domain  $A \mid B$  if and only if  $A \supset B$ .

Let an ideal  $T$  divide the ideal  $M$ . Then the ideal  $T$  is called the *unitary divisor* of  $M$ , if  $(T, \frac{M}{T}) = (1)$ . Here the *greatest common divisor*  $(A, B)$  of two



ideals  $A$  and  $B$  is defined as the ideal  $A + B = \{a + b ; a \in A, b \in B\}$ , i.e. the least (with respect to the set inclusion) ideal containing both ideals  $A$  and  $B$ . Moreover, an ideal  $D$  is called *unitary divisor generated by the divisor  $T$  of  $M$*  provided  $D$  is a unitary divisor of the ideal  $M$  and  $D$  is divisible by exactly the same prime ideals of the ring  $R$  as the ideal  $T$ . We shall denote it by  $D = \langle T \rangle$ .

If (5) is the factorization of an ideal  $M$  with distinct prime ideals  $P_1, \dots, P_r$ ,  $u_i > 0$ ,  $i = 1, \dots, r$ , then given a divisor  $T$  of the ideal  $M$ , define

$$J_T = \{i \in \{1, \dots, r\} ; P_i | T\}.$$

The next result describes the relation between unitary divisors of the ideal  $M$  and idempotents of the residue class ring  $R_M$ .

**Lemma 9** ([LaPo96, Theorem 3.2]) *There exists a one-to-one correspondence between unitary divisors of the ideal  $M$  and idempotents of the residue class ring  $R_M$ . More precisely, every idempotent in  $R_M$  is a solution of the congruence system*

$$\begin{aligned} x &\equiv 0 \pmod{P_i^{u_i}} & \text{for } i \in J_D, \\ x &\equiv 1 \pmod{P_i^{u_i}} & \text{for } i \in \{1, \dots, r\} \setminus J_D, \end{aligned} \quad (6)$$

where  $D$  is a unitary divisor of the ideal  $M$ .

If an idempotent  $[f] \in R_M$  is given by the system (6), where the ideal  $D$  is a unitary divisor of the ideal  $M$ , then we again say that  $[f]$  is the *idempotent belonging to the (unitary) divisor  $D$* .

This implies, for instance, that we have  $2^r$  idempotents in the ring  $R_M$ , and that primitive idempotent  $[e_i]$ , for every  $i = 1, \dots, r$ , is just the idempotent belonging to the unitary divisor

$$M_i = \frac{M}{P_i^{u_i}} = \prod_{\substack{j=1 \\ j \neq i}}^r P_j^{u_j}.$$

This shows that our notation  $J_T$  does not collide with its previous usage. If  $[x] \in R_M$  and  $T = ((x), M)$ , then we say that  $[x]$  *belongs to the divisor  $T$  of  $M$* .

The next result brings us back to FERMAT-EULER Theorem via the explicit determination of  $\nu([x])$ :

**Lemma 10** ([LaPo96, Theorem 4.3]) *Let  $[x] \in R_M$  belong to a divisor  $T = \prod_{j \in J_T} P_j^{v_j}$ , where  $1 \leq v_j \leq u_j$  for every  $j \in J_T$ . Then*

$$\nu([x]) = \begin{cases} 1 & \text{if } T = 1 \quad (J_T = \emptyset), \\ \max_{j \in J_T} \left\lceil \frac{u_j}{v_j} \right\rceil & \text{otherwise.} \end{cases} \quad (7)$$

This Theorem in turn implies that

$$\nu^{(i)} = u_i.$$

For later purposes define the function  $\mathcal{H}$  on proper non-zero ideals  $M$  of a DEDEKIND ring  $R$  by

$$\mathcal{H}^R(M) = \max\{u_i; i \in \{1, \dots, r\}\}$$

if (5) is the decomposition of  $M$  into the product of prime ideals.

If  $[f]$  is the idempotent belonging to the divisor  $D$  of  $M$ , then

$$\nu_{[f]} = \max_{j \in J_D} u_j = \mathcal{H}^R(D);$$

in the case  $[f] = [0]$  we get

$$\nu_{[0]} = \max_{j \in \{1, \dots, r\}} u_j = \nu_{R_M} = \mathcal{H}^R(M).$$

We also have:

**Lemma 11** *Let  $[f]$  be the idempotent of the ring  $R_M$  belonging to the unitary divisor  $D$  of  $M$ . Then*

(i) *The element  $[x]^{\mathcal{H}^R(D)}$  belongs to  $G^{R_M}([f])$  for every  $[x] \in P^{R_M}([f])$ .*

(ii) *The element  $[x]^{\mathcal{H}^R(M)}$  belongs to a group for every  $[x] \in R_M$ .*

*The numbers  $\mathcal{H}^R(D)$  and  $\mathcal{H}^R(M)$  are the least positive integers possessing these properties.*

Of fundamental importance is also the following structural result:

**Lemma 12** *Let  $[f] \in R_M$  be the idempotent belonging to the unitary divisor  $D$  of  $M$ . Then the finite commutative rings  $R_{\frac{M}{D}}$  and  $[f]_M R_M$  with identities  $[1]_{\frac{M}{D}}$  and  $[f]_M$  are isomorphic.*

**Corollary 12.1** *Let  $[f] \in R_M$  be the idempotent belonging to the unitary divisor  $D$  of  $M$ . Then the unit groups  $G^{R_{\frac{M}{D}}}([1]_{\frac{M}{D}})$  and  $G^{[f]_M R_M}([f]_M)$  are isomorphic.*

**Corollary 12.2** *If  $[e_i]$ ,  $i = 1, \dots, r$  are primitive idempotents of  $R_M$ , then*

$$G^{R_M}([e_i]_M) \simeq G^{R_{P_i^*}}([1]_{P_i^*}).$$

This shows that for the determination of the values  $\mu^{(i)}$ ,  $\mu_{[f]}$ , and  $\mu_{R_M} = \mu_{[1]}$  the information about the structure of the groups  $G^{R_{P^*}}([1]_{P^*})$ , where  $P$  is the prime ideal of the ring  $R$  and  $u > 0$ , is necessary. Thus for instance, a classical structural result says:

**Lemma 13** *If  $p$  is a prime number in  $Z$  and  $u > 0$ , then*

$$G^{Z, p^u}([1]_{p^u}) \simeq \begin{cases} Z_1 & \text{if } p = 2, \quad u = 1, \\ Z_2 & \text{if } p = 2, \quad u = 2, \\ Z_2 \times Z_{2^{u-2}} & \text{if } p = 2, \quad u > 2, \\ Z_{p^u - p^{u-1}} & \text{if } p > 2. \end{cases}$$

Therefore the exponent of the unit group  $G^{Z, m}([1]_m)$ , where  $m \in Z$ ,  $m \neq 0$ , is given by the so-called *Carmichael function*  $\lambda$  defined by:

$$\lambda(m) = \begin{cases} 1 & \text{if } m = 1, \\ 2^{u-2} & \text{if } m = 2^u, \quad u > 2, \\ \varphi(m) & \text{if } m = 2, 4, \text{ or } p^u \text{ for odd prime } p, \\ \text{l.c.m.}\{\lambda(p_i^{u_i}); i = 1, \dots, r\} & \text{if } m = p_1^{u_1} \dots p_r^{u_r}, \end{cases} \quad (8)$$

where  $\varphi$  is the EULER totient function, i.e.:

**Lemma 14** *For every  $j = 1, \dots, r$*

$$\mu^{(j)} = \lambda(p_j^{u_j}) = \begin{cases} 1 & \text{if } p_j = 2, \quad u_j = 1, \\ 2 & \text{if } p_j = 2, \quad u_j = 2, \\ 2^{u_j-2} & \text{if } p_j = 2, \quad u_j > 2, \\ p_j^{u_j} - p_j^{u_j-1} & \text{if } p_j > 2. \end{cases}$$

This yields the following (by the way the best possible) extensions of FERMAT-EULER Theorem for  $Z$  which are proved in [Schw81], where

$$H(m) = \mathcal{H}^Z((m)).$$

**Proposition 6 (Global Fermat-Euler Theorem)** *Let  $a, m \in Z$ ,  $m \neq 0$ . If  $\kappa, \delta \in N$  with  $\kappa \geq H(m), \lambda(m) \mid \delta$ , then*

$$a^{\kappa+\delta} \equiv a^\kappa \pmod{m},$$

where  $H(n) = \max\{\alpha_1, \alpha_2, \dots, \alpha_k\}$  for  $n$  having the standard form  $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ . The exponents  $\lambda(m), H(m)$  are the least positive integers for which the congruence is true for every  $a$ .

If again, given a divisor  $d$  of  $m$ ,  $\langle d \rangle$  denotes the unitary divisor of  $m$  having the same set of prime divisors as  $d$ , and, a *unitary divisor* of  $m$  is such a divisor  $t$  of  $m$  for which  $(m, m/t) = 1$ , then

**Proposition 7 (Local Fermat-Euler Theorem)** *Let  $a, m \in Z$ ,  $m \neq 0$  and  $d = \langle (a, m) \rangle$ . If  $\kappa, \delta \in N$  with  $\kappa \geq H(d), \lambda(\frac{m}{d}) \mid \delta$ , then*

$$a^{\kappa+\delta} \equiv a^\kappa \pmod{m},$$

The exponents  $\lambda(m/d), H(d)$  are the least possible positive integers over the set  $P(d) = \{n \in Z : \langle (n, m) \rangle = d\}$ .

Various other forms of FERMAT-EULER Theorem found in the literature can be derived from the just given one using that the LAGRANGE's Theorem of group theory which in case of  $Z_m$  says

$$\forall_{m \in N} \quad \lambda(m) \mid \varphi(m). \quad (9)$$

This follows directly also from (8). For concretization of Propositions 2 and 5 for other rings the values of  $\mu_e$  and  $\mu_R$  are needed. In [LaPo96] the corresponding values for GAUSSIAN integers, and other quadratic extensions of  $Z$  and general number fields can be found.

## 2 Smarandache's algorithm

Given two integers  $a, m$  with  $m \neq 0$ , F.SMARANDACHE [Smar81] proved that the following algorithm terminates

Let	$d_0 = (a, m),$	$a = a_0 d_0,$ $m = m_0 d_0,$	$(a_0, m_0) = 1.$
If $d_0 > 1$ then	$d_1 = (d_0, m_0),$	$d_0 = d_0^1 d_1,$ $m_0 = m_1 d_1,$	$(d_0^1, m_1) = 1.$
If $d_1 > 1$ then	$d_2 = (d_1, m_1),$	$d_1 = d_1^1 d_2,$ $m_1 = m_2 d_2,$	$(d_1^1, m_2) = 1.$
If $d_2 > 1$ then	$d_3 = (d_2, m_2),$	$d_2 = d_2^1 d_3,$ $m_2 = m_3 d_3,$	$(d_2^1, m_3) = 1.$
etc. until $d_{s-1} > 1$ and	$d_s = (d_{s-1}, m_{s-1}),$	$d_{s-1} = d_{s-1}^1 d_s,$ $m_{s-1} = m_s d_s,$	$(d_{s-1}^1, m_s) = 1,$

where  $d_s = 1.$

This algorithm provided him the basis for the following generalization of the FERMAT-EULER Theorem:

**Proposition 8 (Smarandache, [Smar81, Théorème])** *If  $a, m \in Z$ ,  $m \neq 0$ , then*

$$a^{\varphi(m_s)+s} \equiv a^s \pmod{m}, \quad (10)$$

*where  $m_s$  and  $s$  are defined through the above algorithm and  $\varphi$  is the EULER's totient function.*

It follows from the above algorithm that

$$d_s \mid d_{s-1} \mid \dots \mid d_0, \quad d_0 = (a, m),$$

$$\begin{aligned}
& m_s \mid m_{s-1} \mid \dots \mid m_0 \mid m, \\
& (d_i^1, m_s) = 1 \quad \text{for } i = 0, 1, 2, \dots, s-1, \\
& (a, m_s) = 1,
\end{aligned} \tag{11}$$

$$\begin{aligned}
m &= (d_0^1)^1 (d_1^1)^2 \dots (d_{s-1}^1)^s \cdot m_s, \\
a &= a_0 d_0^1 d_1^1 \dots d_{s-1}^1 d_s.
\end{aligned} \tag{12}$$

Relation (11) is employed as the starting point of the SMARANDACHE's proof of the above Proposition 8 through the EULER Theorem

$$a^{\varphi(m_s)} \equiv 1 \pmod{m_s}. \tag{13}$$

However, as we noted in the previous Section of this paper,  $\varphi(m_s)$  is not the best exponent for which (13) is true for every  $a$  coprime to  $m_s$ . The best exponent is given by CARMICHAEL's function  $\lambda(m_s)$  as relations (7) and (9) show. Therefore an immediate check of the SMARANDACHE's proof implies that SMARANDACHE's result of Proposition 8 can be improved to the form:

**Theorem 1** *If  $a, m \in \mathbb{Z}$ ,  $m \neq 0$ , then*

$$a^{\lambda(m_s)+s} \equiv a^s \pmod{m}, \tag{14}$$

where  $m_s$ , and  $\square$  are defined through as above and  $\lambda$  is the Carmichael's function.

### 3 Generalized Smarandache's algorithm

In this Section give another proof of a generalization of Proposition 8 based on the results quoted in Section 1.

$R$  will again denote a residually finite DEDEKIND domain. Here the SMARANDACHE's algorithm acquires the following form:

Given two ideals  $A, M$  with  $M \neq (0)$ , let

Let	$D_0 = (A, M),$	$A = A_0 D_0,$ $M = M_0 D_0,$	$(A_0, M_0) = R.$
If $D_0 \neq R$ then	$D_1 = (D_0, M_0),$	$D_0 = D_0^1 D_1,$ $M_0 = M_1 D_1,$	$(D_0^1, M_1) = R.$
If $D_1 \neq R$ then	$D_2 = (D_1, M_1),$	$D_1 = D_1^1 D_2,$ $M_1 = M_2 D_2,$	$(D_1^1, M_2) = R.$
	$\vdots$		
If $D_{s-1} \neq R$ then	$D_s = (D_{s-1}, M_{s-1}),$	$D_{s-1} = D_{s-1}^1 D_s,$ $M_{s-1} = M_s D_s,$	$(D_{s-1}^1, M_s) = R,$
etc.			

Though we give a more explicit proof of the above SMARANDACHE's result in this more general setting, the original SMARANDACHE's ideas can also be employed here if the well ordering principle of the set of positive integers used by SMARANDACHE over the set

$$1 = d_s \leq d_{s-1} \leq \dots \leq d_0, \quad d_0 = (a, m),$$

is replaced in  $R$  through the norm function  $\mathcal{N}$  over the set

$$1 = \mathcal{N}(\mathcal{D}_f) \leq \mathcal{N}(\mathcal{D}_{f-\infty}) \leq \dots \leq \mathcal{N}(\mathcal{D}_i), \quad \mathcal{D}_i = (\mathcal{A}, \mathcal{M}),$$

by means of the following elementary results:

**Lemma 15** ([Gilm72, Exercise 8, p.467]) *If  $R$  is a Dedekind domain and  $A, B$  are two non-zero ideals with finite norm  $\mathcal{N}(A), \mathcal{N}(B)$ , then  $AB$  also has finite norm and*

$$\mathcal{N}(AB) = \mathcal{N}(A)\mathcal{N}(B). \quad (15)$$

Note that there follows from the subsequent Exercise 9, [Gilm72] that the truth of (15) for every couple of non-zero ideals in a residually finite domain  $R$  forces that  $R$  is DEDEKIND.

**Lemma 16** *Let  $R$  be a residually finite Dedekind domain. If  $A, B$  are two ideals of  $R$  with  $AB$ , then  $\mathcal{N}(A)\mathcal{N}(B)$ .*

*Proof.* As already mentioned if  $R$  is DEDEKIND then  $A \subset B$  holds if and only if there exists an ideal  $C$  such that  $A = BC$ . If  $\mathcal{N}(A) = \mathcal{N}(B)$ , then  $\mathcal{N}(C) = 1$ , i.e.  $C = R$ . Consequently,  $A = BC = B$ , which is impossible due to  $AB$ .

That the SMARANDACHE's algorithm also terminates in this more general setting follows from the next result:

**Theorem 2** *Let  $M = P_1^{\alpha_1} P_2^{\alpha_2} \dots P_k^{\alpha_k}$  and  $A = P_1^{\beta_1} P_2^{\beta_2} \dots P_k^{\beta_k}$  be decompositions of ideals  $M$  and  $A$  into the product of distinct prime ideals of  $R$  with  $0 \leq \alpha_i$  and  $0 \leq \beta_i$  for  $i = 1, 2, \dots, k$ . Then the generalized Smarandache's algorithm terminates for  $s$  given by*

$$s = \max \left\{ 0, \left\lceil \frac{\alpha_i}{\beta_i} \right\rceil : \text{for } i = 1, 2, \dots, k \text{ with } \beta_i \neq 0 \right\}.$$

*Proof.* We shall discuss the contribution of every prime ideal  $P$  separately. Let  $P^\alpha || M$  and  $P^\beta || A$ . If  $\beta \neq 0$ , put

$$\alpha = K\beta + q, \quad 0 < q \leq \beta,$$

and  $K = 0$  if  $\beta = 0$ .

If  $D_0 = (A, M)$  and  $P^{\gamma_0} || D_0$ , then  $\gamma_0 = \min\{\alpha, \beta\}$ . Consequently, if  $M = M_0 D_0$ , then  $P^{\mu_0} || M_0$ , where  $\mu_0 = \alpha - \gamma_0$ . Thus if  $\alpha \leq \beta$  or  $\beta = 0$ , i.e. if  $K = 0$ , then  $P^0 || M_0$ , and  $P$  does not contribute more to the whole process.

If  $K \geq 1$ , then  $\gamma_0 = \beta$  and  $\mu_0 = \alpha - \beta > 0$ , i.e.  $P^{\alpha-\beta} \neq R$ , and we can continue in the SMARANDACHE's algorithm. If  $D_1 = (D_0, M_0)$  and  $P^{\gamma_1} \parallel D_1$ , then  $\gamma_1 = \min\{\gamma_0, \mu_0\}$ . Since  $M_0 = M_1 D_1$ ,  $P^{\mu_1} \parallel M_1$  with  $\mu_1 = \mu_0 - \gamma_1$ . Consequently,  $\mu_1 = 0$  if  $\beta < \alpha \leq 2\beta$ , i.e. if  $K = 1$ , or  $\mu_1 = \alpha - 2\beta$  provided  $K > 1$ . Thus if  $K = 1$ , then  $P^0 \parallel M_1$ , and the contribution of  $P$  terminates. If  $K > 1$  then  $\mu_1 = \alpha - 2\beta$  and  $\gamma_1 = \beta$ , etc.

In the last but one step,  $\mu_{K-1} = \alpha - K\beta$  and  $P^{\gamma_{K-1}} \parallel D_{K-1}$  with  $\gamma_{K-1} = \min\{\gamma_{K-2}, \mu_{K-2}\} = \beta$ . Then  $P^{\gamma_K} \parallel D_K$  implies  $\gamma_K = \min\{\gamma_{K-1}, \mu_{K-1}\} = \alpha - K\beta$  and

$M_{K-1} = M_K D_K$  yields  $\mu_K = \mu_{K-1} - \gamma_{K-1} = 0$ , i.e.  $PM_K$ .

This shows that the SMARANDACHE's algorithm really stops after

$$\max \left\{ 0, \left\lceil \frac{\alpha_i}{\beta_i} \right\rceil : i = 1, 2, \dots, k \text{ with } \beta_i \neq 0 \right\}$$

steps, and the proof is finished.

Lemma 10 immediately then proves:

**Corollary 2.1** *If  $[x] \in R_M$  belongs to the divisor  $T = \prod_{j \in J_T} P_j^{\beta_j}$ , where  $1 \leq \beta_j \leq \alpha_j$  for every  $j \in J_T$  then*

$$\nu([x]) = \begin{cases} 1, & \text{if } T = 1 \text{ (i.e. if } J_T = \emptyset), \\ s, & \text{otherwise.} \end{cases}$$

It follows from the last Corollary that SMARANDACHE's number  $s$  is a more suitable tool for extension of the  $(p-1)$ -power version of FERMAT Theorem, while  $\nu([x])$  does this for its  $p$ -power version.

Moreover we have:

**Theorem 3** *If  $D_0 = (A, M)$  and  $D = \langle D_0 \rangle$ , then*

$$M_s = \frac{M}{D}.$$

*Proof.* Let  $P^\alpha \parallel M$  but  $PD$ . Then  $P^\alpha \parallel M_s$  and  $PD_0$ . Consequently,

$$P^\alpha \parallel M_i, \quad i = 0, 1, \dots, s,$$

i.e.  $P^\alpha \parallel M_s$ .

Let  $P \mid M$  and also  $P \mid D$ . We claim that  $PM_s$ . In the opposite case

$$P \mid M_s \mid M_{s-1} \mid \dots \mid M_0 \mid M,$$

and simultaneously  $P \mid D_0$ . Therefore  $P \mid D_1 = (D_0, M_0)$ , and thus  $P \mid D_2 = (D_1, M_1)$ , etc.,  $P \mid D_s = (D_{s-1}, M_{s-1})$ . A contradiction, since  $D_s = 1$ .

This together with Lemma 8 gives the following extension of SMARANDACHE's contribution to the individual type FERMAT-EULER Theorem to residually finite DEDEKIND domains:

**Theorem 4** *Let  $R$  be a residually finite Dedekind domain and  $M$  its non-zero ideal. Then given an element  $a \in R$ , let  $s, M_s$  be determined by the above Smarandache's algorithm for  $A = (a)$ , and  $M$ . Then*

$$a^{\varphi_R(M_s)+s} \equiv a^s \pmod{M}. \quad (16)$$

It follows from the above discussion that the exponent  $\varphi_R(M_s)$  is not the best possible. The best one is given by the order of the cyclic group  $C(a)$  in  $R_M$ . The “next” best exponent is given by the exponent of the maximal subgroup of the multiplicative semigroup of  $R_M$  belonging to the idempotent belonging to the unitary divisor  $D = \langle (a), M \rangle$ . In the case when  $R = \mathbb{Z}$  this is given through the CARMICHAEL function. The reader is again referred to [LaPo96] for how the corresponding values can be computed in the case of algebraic number fields. The necessary facts can also be found in [Naka79]. For other residually finite commutative rings the corresponding numbers can be computed using (3) and Lemma 12 and its Corollaries 12.1, 12.2.

## 4 Applications

As noticed by SMARANDACHE in [Smar81] his algorithm can be easily implemented. Namely:

Step 1:  $A := a, M := m, i := 0$

Step 2: COMPUTE  $d = (a, m)$  AND  $M' = M/d$

Step 3: IF  $d = 1$  THEN  $s = i$  and  $m_s = M'$ ; STOP

Step 4: IF  $d \neq 1$  THEN  $A := d, M := M', i := i + 1$ ; GOTO 2

In conjunction with the above given form of individual FERMAT-EULER Theorem the SMARANDACHE's algorithm can be used for a effective determinations of:

- The highest power in which a prime from a given set  $\{p_1, p_2, \dots, p_k\}$  of primes divides a given integer  $n$ . Simply apply the above algorithm with  $a = p_1 \dots p_k$  and  $m = n$ .
- the least power  $k$  for which a given number  $x$  belongs to a subgroup of the multiplicative semigroup of  $\mathbb{Z}_n$ , the residues modulo  $n$ . Again apply the the algorithm with  $a = x$ , and  $m = n$ .

Adaptation of the above ideas to other residually finite rings along above lines is left to the reader.



## 5 Acknowledgement

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# A few Smarandache Integer Sequences

Henry Ibstedt

## Abstract

This paper deals with the analysis of a few Smarandache Integer Sequences which first appeared in *Properties of the Numbers*, F. Smarandache, University of Craiova Archives, 1975. The first four sequences are recurrence generated sequences while the last three are concatenation sequences.

*The Non-Arithmetic Progression:*  $\{a_i : a_i \text{ is the smallest integer such that } a_i > a_{i-1} \text{ and such that for } k \leq i \text{ there are at most } t-1 \text{ equal differences } a_k - a_{k_1} = a_{k_1} - a_{k_2} = \dots = a_{k_{t-2}} - a_{k_{t-1}}\}$

A strategy for building a  $t$ -term non-arithmetic progression is developed and computer implemented for  $3 \leq t \leq 15$  to find the first 100 terms. Results are given in tables and graphs together with some observations on the behaviour of these sequences.

*The prime-Product Sequence:*  $\{t_n : t_n = p_n \# + 1, p_n \text{ is the } n\text{th prime number}\}$ , where  $p_n \#$  denotes the product of all prime numbers which are less than or equal to  $p_n$ .

The number of primes  $q$  among the first 200 terms of the prime-product sequence is given by  $6 \leq q \leq 9$ . The six confirmed primes are terms numero 1, 2, 3, 4, 5 and 11. The three terms which are either primes or pseudo primes (according to Fermat's little theorem) are terms numero 75, 171 and 172. The latter two are the terms  $1019\# + 1$  and  $1021\# + 1$ .

*The Square-Product Sequence:*  $\{t_n : t_n = (n!)^2 + 1\}$

As in the previous sequence the number of primes in the sequence is of particular interest. Complete prime factorization was carried out for the first 37 terms and the number of prime factors  $f$  was recorded. Terms 38 and 39 are composite but were not completely factorized. Complete factorization was obtained for term no 40. The terms of this sequence are in general much more time consuming to factorize than those of the prime-product sequence which accounts for the more limited results. Using the same method as for the prime-product sequence the terms  $t_n$  in the interval  $40 < n \leq 200$  which may possible be primes were identified. There are only two of them, term #65:  $N = (65!)^2 + 1$  which is a 182 digit number and term #76:  $N = (76!)^2 + 1$  which has 223 digits.

*The Prime-Digital Sub-Sequence:* The prime-digital sub-sequence is the set  $\{M = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \dots + a_k \cdot 10^k : M \text{ is a prime and all digits } a_0, a_1, a_2, \dots, a_k \text{ are primes}\}$

A proof is given for the theorem: The Smarandache prime-digital sub sequence is infinite, which until now has been a conjecture.

*Smarandache Concatenated Sequences:* Let  $G = \{g_1, g_2, \dots, g_k, \dots\}$  be an ordered set of positive integers with a given property  $G$ . The corresponding concatenated  $S.G$  sequence is defined through  $S.G = \{a_i : a_1 = g_1, a_k = a_{k-1} \cdot 10^{1+\log_{10} g_k} + g_k, k \geq 1\}$ .

*The S.Odd Sequence:* Fermat's little theorem was used to find all primes/pseudo-primes among the first 200 terms. There are only five cases which all were confirmed to be primes using the elliptic curve prime factorization program, the largest being term 49:

135791113151719212325272931333537394143454749515355575961636567697173757779818385878991939597

Term #201 is a 548 digit number.

*The S.Even Sequence:* The question how many terms are  $n$ th powers of a positive integer was investigated. It was found that there is not even a perfect square among the first 200 terms of the sequence. Are there terms in this sequence which are  $2 \cdot p$  where  $p$  is a prime (or pseudo prime)? Strangely enough not a single term was found to be of the form  $2 \cdot p$ .

*The S.Prime Sequence:* How many are primes? Again we apply the method of finding the number of primes/pseudo primes among the first 200 terms. Terms #2 and #4 are primes, namely 23 and 2357. There are only two other cases which are not proved to be composite numbers: term #128 which is a 355 digit number and term #174 which is a 499 digit number.

## I. The Non-Arithmetic Progression

This integer sequence was defined in simple terms in the February 1997 issue of Personal Computer World. It originates from the collection of Smarandache Notions. We consider an ascending sequence of positive integers  $a_1, a_2, \dots, a_n$  such that each element is as small as possible and no  $t$ -term arithmetic progression is in the sequence. In order to attack the problem of building such sequences we need a more operational definition.

**Definition:** The  $t$ -term non-arithmetic progression is defined as the set :

$\{a_i : a_i \text{ is the smallest integer such that } a_i > a_{i-1} \text{ and such that for } k \leq i \text{ there are at most } t-1 \text{ equal differences } a_k - a_{k_1} = a_{k_1} - a_{k_2} = \dots = a_{k_{i-2}} - a_{k_{i-1}}\}$

From this definition we can easily formulate the starting set of a  $t$ -term non-arithmetic progression:

$$\{1, 2, 3, \dots, t-1, t+1\} \text{ or } \{a_i : a_i = i \text{ for } i \leq t-1 \text{ and } a_t = t+1 \text{ where } t \geq 3\}$$

It may seem clumsy to bother to express these simple definitions in stringent terms but it is in fact absolutely necessary in order to formulate a computer algorithm to generate the terms of these sequences.

**Question:** How does the density of a  $t$ -term non arithmetic progression vary with  $t$ , i.e. how does the fraction  $a_k/k$  behave for  $t \geq 3$ ?<sup>1</sup>

**Strategy for building a  $t$ -term non-arithmetic progression:** Given the terms  $a_1, a_2, \dots, a_k$  we will examine in turn the following candidates for the term  $a_{k+1}$ :

$$a_{k+1} = a_k + d, d=1, 2, 3, \dots$$

Our solution is the smallest  $d$  for which none of the sets

$$\{a_1, a_2, \dots, a_k, a_k+d, a_k+d-e, a_k+d-2e, \dots, a_k+d-(t-1)e : e \geq d\}$$

contains a  $t$ -term arithmetic progression.

We are certain that  $a_{k+1}$  exists because in the worst case we may have to continue constructing sets until the term  $a_k+d-(t-1)e$  is less than 1 in which case all possibilities have been tried with no  $t$  terms in arithmetic progression. The method is illustrated with an example in diagram 1.

In the computer application of the above method the known terms of a no  $t$ -term arithmetic progression were stored in an array. The trial terms were in each case added to this array. In the example we have for  $d=1, e=1$  the array: 1,2,3,5,6,8,9,10,11,10,9,8. The terms are arranged in ascending order: 1,2,3,5,6,8,8,9,9,10,10,11. Three terms 8,9 and 10 are duplicated and 11 therefore has to be rejected. For  $d=3, e=3$  we have 1,2,3,5,6,8,9,10,13,10,7,4 or in ascending order: 1,2,3,4,5,6,7,8,9,10,10,13 this is acceptable but we have to check for all values of  $d$  that produce terms

<sup>1</sup> This question is slightly different from the one posed in the Personal Computer World where also a wider definition of a  $t$ -term non arithmetic progression is used in that it allows  $a_2 > a_1$  to be chosen arbitrarily.

which may form a 4-term arithmetic progression and as we can see from diagram 1 this happens for  $d=3$ ,  $e=4$ , so 13 has to be rejected. However, for  $d=5$ ,  $e=5$  no 4-term arithmetic progression is formed and  $e=6$  does not produce terms that need to be checked, hence  $a_9 = 15$ .

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
Known terms		1	2	3		5	6		8	9	10						
Trials																	
d=1	e=1								8	9	10	11					reject 11
d=2	e=2						6		8		10		12				reject 12
d=3	e=3				4			7			10			13			try next e
	e=4	1				5				9				13			reject 13
d=4	e=3		2				6				10				14		reject 14
d=5	e=5					5					10					15	accept 15

Diagram 1. To find the 9<sup>th</sup> term of the 4-term non-arithmetic progression.

Routines for ordering an array in ascending order and checking for duplication of terms were included in a QBASIC program to implement the above strategy.

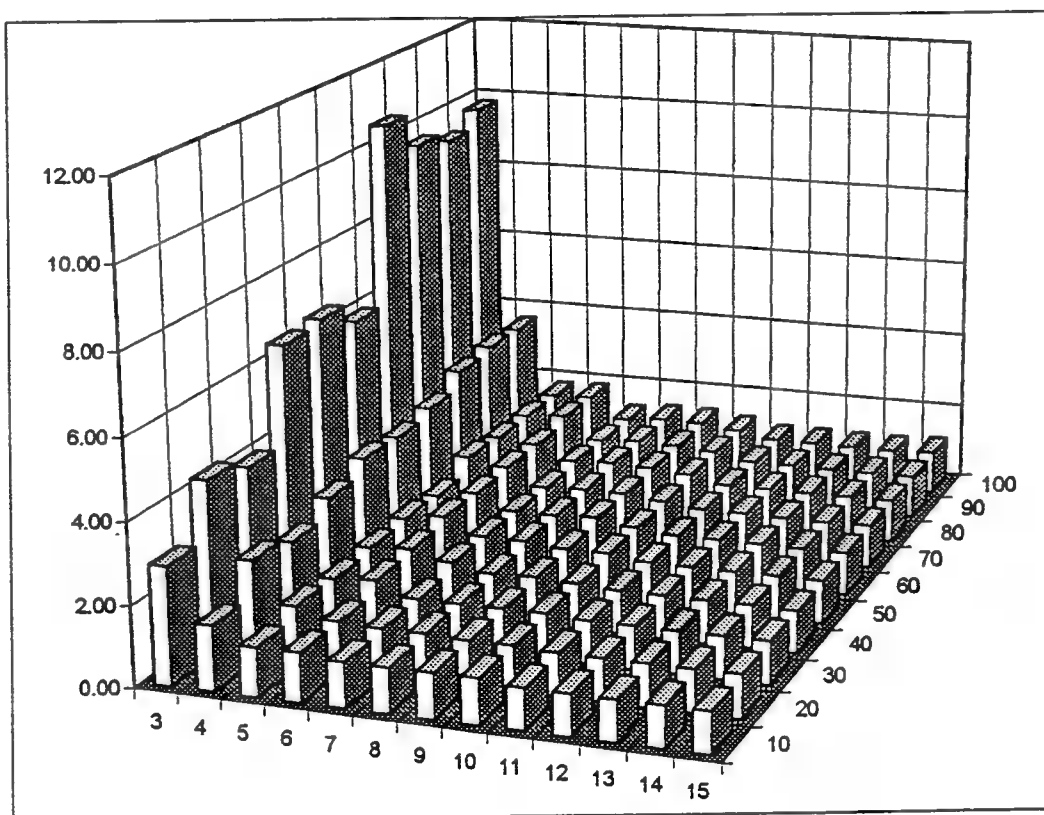


Diagram 2.  $a_k/k$  for non-arithmetic progressions with  $t=3, 4, 5, \dots, 15$ . Bars are shown for  $k = \text{multiples of } 10$ .

**Results and observations:** Calculations were carried out for  $3 \leq t \leq 15$  to find the first 100 terms of each sequence. The first 65 terms and the 100<sup>th</sup> term are shown in table 1. In diagram 2 the fractions  $a_k/k$  has been chosen as a measure of the density of these sequences. The looser the terms are packed the larger is  $a_k/k$ . In fact for  $t > 100$  the value of  $a_k/k = 1$  for the first 100 terms.

In table 1 there is an interesting leap for  $t=3$  between the 64<sup>th</sup> and the 65<sup>th</sup> terms in that  $a_{64} = 365$  and  $a_{65} = 730$ . Looking a little closer at such leaps we find that:

Table 1. The 65 first terms of the non-arithmetic progressions for  $t=3$  to 15.

#	t=3	t=4	t=5	t=6	t=7	t=8	t=9	t=10	t=11	t=12	t=13	t=14	t=15
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	4	3	3	3	3	3	3	3	3	3	3	3	3
4	5	5	4	4	4	4	4	4	4	4	4	4	4
5	10	6	6	5	5	5	5	5	5	5	5	5	5
6	11	8	7	7	6	6	6	6	6	6	6	6	6
7	13	9	8	8	8	7	7	7	7	7	7	7	7
8	14	10	9	9	9	9	8	8	8	8	8	8	8
9	28	15	11	10	10	10	10	9	9	9	9	9	9
10	29	16	12	12	11	11	11	11	10	10	10	10	10
11	31	17	13	13	12	12	12	12	12	11	11	11	11
12	32	19	14	14	13	13	13	13	13	13	12	12	12
13	37	26	16	15	15	14	14	14	14	14	14	13	13
14	38	27	17	17	16	16	15	15	15	15	15	15	14
15	40	29	18	18	17	17	16	16	16	16	16	16	16
16	41	30	19	19	18	18	17	17	17	17	17	17	17
17	82	31	26	20	19	19	19	18	18	18	18	18	18
18	83	34	27	22	20	20	20	20	19	19	19	19	19
19	85	37	28	23	22	21	21	21	20	20	20	20	20
20	86	49	29	24	23	23	22	22	21	21	21	21	21
21	91	50	31	25	24	24	23	23	23	22	22	22	22
22	92	51	32	26	25	25	24	24	24	24	23	23	23
23	94	53	33	33	26	26	27	25	25	25	24	24	24
24	95	54	34	34	27	27	28	26	26	26	25	25	25
25	109	56	36	35	29	28	29	27	27	27	27	26	26
26	110	57	37	36	30	30	30	28	28	28	28	28	27
27	112	58	38	37	31	31	31	31	29	29	29	29	28
28	113	63	39	39	32	32	32	32	30	30	30	30	29
29	118	65	41	43	33	33	33	33	31	31	31	31	31
30	119	66	42	44	34	34	34	34	32	32	32	32	32
31	121	67	43	45	36	35	37	35	34	33	33	33	33
32	122	80	44	46	37	37	38	36	35	35	34	34	34
33	244	87	51	47	38	38	39	37	36	36	35	35	35
34	245	88	52	49	39	39	40	38	37	37	36	36	36
35	247	89	53	50	40	40	41	39	38	38	37	37	37
36	248	91	54	51	41	41	43	41	39	39	38	38	38
37	253	94	56	52	50	42	44	42	40	40	40	39	39
38	254	99	57	59	51	44	45	43	41	41	41	41	40
39	256	102	58	60	52	45	46	44	42	42	42	42	41
40	257	105	59	62	53	46	47	45	43	43	43	43	42
41	271	106	61	63	54	47	48	49	45	44	44	44	45
42	272	109	62	64	55	48	49	50	46	46	45	45	46
43	274	110	63	65	57	49	50	51	47	47	46	46	47
44	275	111	64	66	58	50	53	52	48	48	47	47	48
45	280	122	66	68	59	59	55	53	49	49	48	48	49
46	281	126	67	69	60	60	56	54	50	50	49	49	50
47	283	136	68	71	61	61	57	55	51	51	50	50	51
48	284	145	69	73	62	62	58	58	52	52	51	51	52
49	325	149	76	77	64	63	59	59	53	53	53	52	53
50	326	151	77	85	65	64	60	60	54	54	54	54	54
51	328	152	78	87	66	65	64	61	56	55	55	55	55
52	329	160	79	88	67	67	65	62	57	57	56	56	56
53	334	163	81	89	68	69	66	63	58	58	57	57	58
54	335	167	82	90	69	70	67	64	59	59	58	58	59
55	337	169	83	91	71	71	68	65	60	60	59	59	60
56	338	170	84	93	72	72	69	66	61	61	60	60	61
57	352	171	86	96	73	74	70	68	62	62	61	61	62
58	353	174	87	97	74	75	71	69	63	63	62	62	63
59	355	176	88	98	75	76	78	70	64	64	63	63	64
60	356	177	89	99	76	77	79	71	65	65	64	64	65
61	361	183	91	100	78	78	80	72	67	66	66	65	66
62	362	187	92	103	79	79	81	73	68	68	67	67	67
63	364	188	93	104	80	81	82	74	69	69	68	68	68
64	365	194	94	107	81	84	83	75	70	70	69	69	69
65	730	196	126	111	82	85	84	77	71	71	70	70	70
...													
100	977	360	179	183	130	139	138	126	109	109	108	108	113

Leap starts at		Leap finishes at	
5		10	
14	$=3 \cdot 5 - 1$	28	$=2 \cdot 14$
41	$=3 \cdot 14 - 1$	82	$=2 \cdot 41$
122	$=3 \cdot 41 - 1$	244	$=2 \cdot 122$
365	$=3 \cdot 122 - 1$	730	$=2 \cdot 365$

Does this chain of regularity continue indefinitely?

Sometimes it is easier to look at what is missing than to look at what we have. Here are some observations on the only excluded integers when forming the first 100 terms for  $t=11, 12, 13$  and  $14$ .

For $t=11$ : 11, 22, 33, 44, 55, 66, 77, 88, 99	The $n$ th missing integer is $11 \cdot n$
For $t=12$ : 12, 23, 34, 45, 56, 67, 78, 89, 100	The $n$ th missing integer is $11 \cdot n + 1$
For $t=13$ : 13, 26, 39, 52, 65, 78, 91, 104	The $n$ th missing integer is $13 \cdot n$
For $t=14$ : 14, 27, 40, 53, 66, 79, 92, 105	The $n$ th missing integer is $13 \cdot n + 1$

Do these regularities of missing integers continue indefinitely? What about similar observations for other values of  $t$ ?

## II. The Prime-Product Sequence

The prime-product sequence originates from Smarandache Notions. It was presented to readers of the Personal Computer World's Numbers Count Column in February 1997.

**Definition:** The terms of the prime-product sequence are defined through  $\{t_n : t_n = p_n\# + 1, p_n \text{ is the } n\text{th prime number}\}$ , where  $p_n\#$  denotes the product of all prime numbers which are less than or equal to  $p_n$ .

The sequence begins  $\{3, 7, 31, 211, 2311, 30031, \dots\}$ . In the initial definition of this sequence  $t_1$  was defined to be equal to 2. However, there seems to be no reason for this exception.

**Question:** How many members of this sequence are prime numbers?

The question is in the same category as questions like '*How many prime twins are there? How many Carmichael numbers are there?, etc.*' So we may have to contend ourselves by finding how frequently we find prime numbers when examining a fairly large number of terms of this sequence.

From the definition it is clear that the smallest prime number which divides  $t_n$  is larger than  $p_n$ . The terms of this sequence grow rapidly. The prime number functions  $prmdiv(n)$  and  $nextprm(n)$  built into the *Ubasic* programming language were used to construct a prime factorization program for  $n < 10^{19}$ . This program was used to factorize the 18 first terms of the sequence. An elliptic curve factorization program, ECM.UB, conceived by Y. Kida was adapted to generate and factorize further terms up to and including the 49th term. The result is shown in table 2. All terms analysed were found to be square free. A scatter diagram, Diagram 3, illustrates how many prime factors there are in each term.

The 50th term presented a problem.  $t_{50} = 126173 \cdot n$ , where  $n$  has at least two factors. At this point prime factorization begins to be too time consuming and after a few more terms the numbers will be too large to handle with the above mentioned program. To obtain more information the method of factorizing was given up in favor of using Fermat's theorem to eliminate terms which are definitely not prime numbers. We recall Fermat's little theorem:

If  $p$  is a prime number and  $(a, p)=1$  then  $a^{p-1} \equiv 1 \pmod{p}$ .

$a^{n-1} \equiv 1 \pmod{n}$  is therefore a necessary but not sufficient condition for  $n$  to be a prime number. If  $n$  fills the congruence without being a prime number then  $n$  is called a pseudo prime to the base  $a$ ,  $\text{psp}(a)$ . We will proceed to find all terms in the sequence which fill the congruence

$$a^{t_n-1} \equiv 1 \pmod{t_n}$$

for  $50 \leq n \leq 200$ .  $t_{200}$  is a 513 digit number so we need to reduce the powers of  $a$  to the modulus  $t_n$  gradually as we go along. For this purpose we write  $t_n-1$  to the base 2:

$$t_n-1 = \sum_{k=1}^m \delta(k) \cdot 2^k, \text{ where } \delta(k) \in \{0,1\}$$

From this we have

$$a^{t_n-1} = \prod_{k=1}^m a^{\delta(k) \cdot 2^k}$$

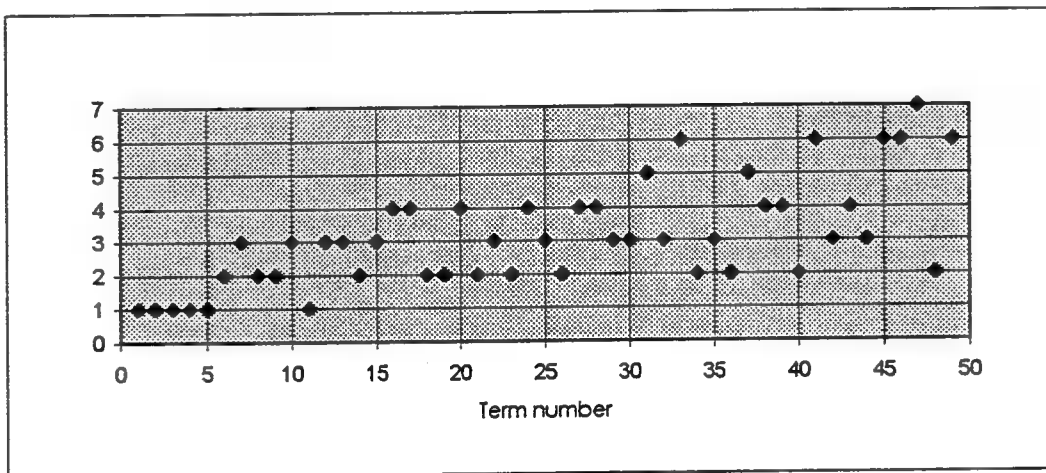


Diagram 3. The number of prime factors in the first 49 terms of the prime-product sequence.

This product expression for  $a^{t_n-1}$  is used in the following *Ubasic* program to carry out the reduction of  $a^{t_n-1}$  modulus  $t_n$ . Terms for which  $\delta(k)=0$  are ignored in the expansion where the exponents  $k$  are contained in the array  $E\%( )$ . The residue modulus  $t_n$  is stored in  $F$ . In the program below the reduction is done to base  $A=7$ .

```

100 dim E%(1000)
110 M=N-1:1%=0
120 T=1:J%=0
130 while (M-T)>=0
140 inc J%:T=2*T
150 wend
160 dec J%:M=M-T\2:inc 1%:E%(1%)=J%
170 if M>0 then goto 120
180 F=1
190 for J%=1 to 1%
200 A=7
210 for K%=1 to E%(J%)
240 A=(A^2)@N
250 next
260 F=F*A:F=F@N
270 next

```

Table 2. Prime factorization of prime-product terms

#	P	L	N=p#+1 and its factors
1	2	1	3
2	3	1	7
3	5	2	31
4	7	3	211
5	11	4	2311
6	13	5	30031 = 59 · 509
7	17	6	510511 = 19 · 97 · 277
8	19	7	9699691 = 347 · 27953
9	23	9	223092871 = 317 · 703763
10	29	10	6469693231 = 331 · 571 · 34231
11	31	12	200540490131
12	37	13	7420738134811 = 181 · 60611 · 676421
13	41	15	304250263527211 = 61 · 450451 · 11072701
14	43	17	13082761331670031 = 61 · 450451 · 11072701
15	47	18	614889782588491411 = 953 · 46727 · 13808181181
16	53	20	32589158477190044731 = 73 · 139 · 173 · 18546761860301
17	59	22	1922760350154212639071 = 277 · 3467 · 105229 · 19026377261
18	61	24	117288381359406970983271 = 223 · 525956867082542470777
19	67	25	7858321551080267055879091 = 54730729297 · 143581524529603
20	71	27	557940830126698960967415391 = 1063 · 303049 · 598841 · 2892214489673
21	73	29	40729680599249024150621323471 = 2521 · 16156160491570418147804951
22	79	31	3217644767340672907899084554131 = 22093 · 1503181961 · 96888414202798247
23	83	33	267064515689275851355624017992791 = 265739 · 1004988035964897329167431269
24	89	35	23768741896345550770650537601358311 = 131 · 1039 · 2719 · 64225891884294373371806141
25	97	37	2305567963945518424753102147331756071 = 2336993 · 13848803 · 71237436024091007473549
26	101	39	23286236435849736090063316880507363071 = 960703 · 242387464553038099079594127301057
27	103	41	23984823528925228172706521638692258396211 = 2297 · 9700398839 · 179365737007 · 6001315443334531
28	107	43	2566376117594999414479597815340071648394471 = 149 · 13203797 · 30501264491063137 · 42767843651083711
29	109	45	279734996817854936178276161872067809674997231 = 334507 · 1290433 · 648046444234299714623177554034701
30	113	47	31610054640417607788145206291543662493274684991 = 5122427 · 2025436786007 · 3046707595069540247157055819
31	127	49	4014476939333036189094441199026045136445885247731 = 1543 · 49999 · 552001 · 57900988201093 · 1628080529999073967231
32	131	51	525896479052627740771371797072411912900610967452631 = 1951 · 22993 · 11723231859473014144932345466415143728266617
33	137	53	72047817630210000485677936198920432067383702541010311 = 881 · 1657 · 32633677 · 160823938621 · 5330099340103 · 1764291759303233
34	139	56	10014646650599190067509233131649940057366334653200433091 = 678279959005528882498681487 · 14764768614544245139224580493
35	149	58	1492182350939279320058875736615841068547583863326864530411 = 87549524399 · 65018161573521013453 · 262140076844134219184937113
36	151	60	225319534991831177328890236228992001350685163362356544091911 = 23269084799180847 · 9683213481319911991636641541802024271084713
37	157	62	35375166993717494840635767087951744212057570647889977422429871 = 1381 · 1867 · 8311930927 · 38893867968570583 · 42440201875440880489113304753
38	163	64	5766152219975951659023630035336134306565384015606066319856068811 = 1361 · 214114727210560829 · 32267019267402210517 · 613228865630544238382107
39	167	66	962947420735983927056946215901134429196419130606213075415963491271 = 205590139 · 53252429177 · 7064576339566763 · 12450154709928940906197946067239
40	173	69	166589903787325219380851695350896256250980509594874862046961683989711 = 62614127 · 2660580156093611580352333193927566158528098772260689062181793
41	179	71	29819592777931214269172453467810429868925511217482600306406141434158091 = 601 · 1651781 · 8564177 · 358995947 · 1525310189119 · 640532866409618954809029861252251
42	181	73	5397346292805549782720214077673687806275517530364350655459511599582614291 = 107453 · 5634838141 · 8914157280964191123344891396571257163632974628403174028667
43	191	76	1030893141925860008499560888835674370998623848299590975192766715520279329391 = 32999 · 175603474759 · 77148541513247 · 230596146643732395958530415862423316227152033
44	193	78	198962376391690981640415251545285153602734402721821058212203976095413910572271 = 21639496447 · 7979125905967339495018877 · 1152307771625979758044020162101777453615909
45	197	80	39195588149163123383161804554421175259738677336198748467804183290796540382737191 = 521831 · 50257723 · 1601684368321 · 39081170243262541027 · 23875913958369977158572653160969521
46	199	82	7799922041683461553249199106329813876687996789903530945093032474868511536164700811 = 467 · 10723 · 57622771 · 5876645549 · 9458145520867 · 486325954430626096097192220405214947865503847
47	211	85	1645783550795210387735581011435590727981167322669649249414629852197255934130751870911 = 1051 · 2179 · 16333 · 43283699 · 75311908487 · 292812710684839 · 46096596672866469293430334044872907384889
48	223	87	367009731827331916465034565550136732339800312955331782619462457039988073311157667212931 = 13867889468159 · 26464714235716608676791598492896703564888100036053342930619468037572880509
49	227	89	83311209124804345037562846379881038241134671040860314654617977748077292641632790457335111 = 3187 · 31223 · 1737142793 · 11463039340315601 · 973104505470446969309113 · 43206785807567189232875099500379

This program revealed that there are at most three terms  $t_n$  of the sequence in the interval  $50 \leq n \leq 200$  which could be prime numbers. These are:

Term #75.  $N=379\#+1$ .  $N$  is a 154 digit number.

$N=1719620105458406433483340568317543019584575635895742560438771105058321655238562613083979651479555788009994557822024565226932906295208262756822275663694111$



Term #171.  $N=1019\# + 1$ .  $N$  is a 425 digit number.

$N=204040689930163741945424641727746076956597971174231219132271310323390261691759299022444537574$   
 $104687288429298622716055678188216854906766619853898399586228024659868813761394041383761530961031$   
 $408346655636467401602797552123175013568630036386123906616684062354223117837423905105265872570265$   
 $003026968347932485267343058016341659487025063671767012332980646166635537169754290487515755971504$   
 $17381063934255689124486029492908966644747931$

Term #172.  $N=1021\# + 1$ .  $N$  is a 428 digit number.

$N=208325544418697180526278559204028744572686528568890074734049007840181457187286244301915872863$   
 $160885721486313893793092847430169408859808718870830265977538813177726058850383316252820523111213$   
 $067921935404833217036456300717761688853571267150232508655634427663661803312009807112476455894240$   
 $568090534683239067457957262234684834336252590008874119591973239736134883450319130587753586846905$   
 $76146066276875058596100236112260054944287636531$

The last two primes or pseudo primes are remarkable in that they are generated by the prime twins 1019 and 1021.

**Summary of results:** The number of primes  $q$  among the first 200 terms of the prime-product sequence is given by  $6 \leq q \leq 9$ . The six confirmed primes are terms numero 1, 2, 3, 4, 5 and 11. The three terms which are either primes or pseudo primes are terms numero 75, 171 and 172. The latter two are the terms  $1019\# + 1$  and  $1021\# + 1$ .

### III. The Square-Product Sequence

**Definition:** The terms of the square-product sequence are defined through  $\{t_n : t_n = (n!)^2 + 1\}$

This sequence has a structure which is similar to the prime-product sequence. The analysis is therefore carried out almost identically to the one done for the prime-product sequence. We merely have to state the results and compare them.

The sequence begins  $\{2, 5, 37, 577, 14401, 518401, \dots\}$

As for the prime-product sequence the question of how many are prime numbers has been raised and we may never know. There are similarities between these two sequences. There are quite a few primes among the first terms. After that they become more and more rare. Complete factorization of the 37 first terms of the square-product sequence was obtained and has been used in diagram 4 which should be compared with the corresponding diagram 3 for the prime-product sequence.

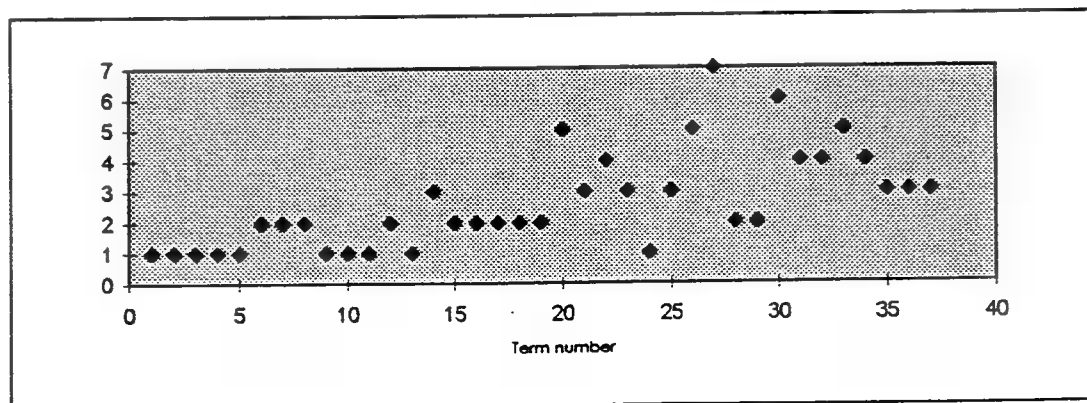


Diagram 4. The number of prime factors in the first 40 terms of the square-product sequence.

n	L	f	N=(n!)2+1 and its factors
1	1	1	2
2	1	1	5
3	2	1	37
4	3	1	577
5	5	1	14401
6	6	2	518401=13·39877
7	8	2	25401601= 101· 251501
8	10	2	1625702401= 17· 95629553
9	12	1	131681894401
10	14	1	13168189440001
11	16	1	1593350922240001
12	18	2	229442532802560001=101· 2271708245569901
13	20	1	38775788043632640001
14	22	3	7600054456551997440001= 29· 109· 2404319663572286441
15	25	2	1710012252724199424000001=1344169·1272170577304043929
16	27	2	437763136697395052544000001=149· 2938007628841577533852349
17	30	2	126513546505547170185216000001= 9049·13980942259426143240713449
18	32	2	40990389067797283140009984000001= 37·1107848353183710355135404792973
19	35	2	14797530453474819213543604224000001=710341·20831587158104092560535861261
20	37	5	59190121813899276854174416896000000001=41·10657· 86816017· 348046955609·448324749841
21	40	3	26102843719929581092690917851136000000001= 61·157·272557624725170524096177486176631513
22	43	4	1263377636044591724886240423994982400000001= 337· 8017· 514049836440277481· 909674823323537849
23	45	3	6683267694675890224648211842933456896000000001= 509·15448374629· 84994002604532747687401741723441
24	48	1	384956219213331276939737002152967117209600000001
25	51	3	24059763700833204808733562634560448256000000000001= 941· 813769831908479758733· 313425331349331290243399417
26	54	5	1626440026176324645070388834096286070210560000000000001= 53· 53· 418633· 6017159668589· 22985889712876096222556462301797
27	57	7	1185674779082540664256313460056192545183498240000000000001= 113· 42461· 745837· 2460281· 7566641· 15238649· 116793504008451126962009
28	59	2	929569026800711882344949752684054955423862620160000000000001= 2122590346576634509· 43794085292997939303952241474982753464389
29	62	2	78176755153998693052102742007290217511468463554560000000000001= 171707860473207588349837· 455289320701414063716469396531758248773
30	65	6	70359076385458823746892467806561195760321617199104000000000000001= 61· 1733· 15661· 359525849· 100636381126568690110069· 1174592249518207759537897
31	68	4	676150755326425929620763661562105309125669074128338944000000000000001= 353· 422041· 13400767181· 33867608180948409085305820793832191570324667821677
32	71	4	692378373454260151931661989439595836544685131907419078656000000000000001= 10591621681· 6415450838021· 522303293914660001204969· 1950882388585355532025429
33	74	5	754000048691689305453579906499719865997162108647179376656384000000000000001= 37· 3121· 4421· 4073332882845936253· 3625813512324480427387450762108578223148052301
34	77	4	87162405628759283710433837191367616509271939759613935941477994000000000000001= 193· 13217· 866100731693· 39452143443145645231476894644096901291197410624286816576197
35	81	3	106773946895230122545281450539425302238581262055270715283105382400000000000000001= 317· 373· 90301965388680848897828545563235360863474820117616321989077716189815715361
36	84	3	13837903517621823881868475992501522797012013156236308470069045755904000000000000001= 73· 57986941373· 32690174316982778045052076286249517817882480631366146754011637734149869
37	87	3	18944089915624276894277943633734584709109446010887506295524523638325760000000000000001= 127406364297881· 49105571194338128021910109· 30279720114524038428292430769814272052327643069
38	90	4+	2735526583816145583533735060711274031995404003972155909073741213591823974400

#### IV. The Smarandache Prime-Digital Sub-Sequence

**Definition:** The prime-digital sub-sequence is the set  $\{M=a_0+a_1 \cdot 10+a_2 \cdot 10^2+\dots+a_k 10^k : M \text{ is a prime and all digits } a_0, a_1, a_2, \dots, a_k \text{ are primes}\}$

The first terms of this sequence are  $\{2, 3, 5, 7, 23, 37, 53, 73, \dots\}$ . Sylvester Smith [1] conjectured that this sequence is infinite. In this paper we will prove that this sequence is in fact infinite. Let's first calculate some more terms of the sequence and at the same time find how many terms there is in the sequence in a given interval, say between  $10^k$  and  $10^{k+1}$ . The program below is written in *Ubasic*. One version of the program has been used to produce table 4 showing the first 100 terms of the sequence. The output of the actual version has been used to produce the calculated part of table 5 which we are going to compare with the theoretically estimated part in the same table.

##### Ubasic program

```

10 point 2
20 dim A%(6),B%(4)
30 for I%=1 to 6:read A%(I%):next
40 data 1,4,6,8,9,0 'Digits not allowed stored in A%()
50 for I%=1 to 4:read B%(I%):next
60 data 2,3,5,7 'Digits allowed stored in B%()
70 for K%=1 to 7 'Calc. for 7 separate intervals
80 M%=0:N=0
90 for E%=1 to 4 'Only 2,3,5 and 7 allowed as first digit
100 P=B%(E%)*10^K%:P0=P:S=(B%(E%)+1)*10^K%:gosub 150
110 next
120 print K%,M%,N,M%/N
130 next
140 end
150 while P<S
160 P=nxtprm(P):P$=str(P) 'Select prime and convert to string
170 inc N 'Count number of primes
180 L%=len(P$):C%=0 'C% will be set to 1 if P not member
190 for I%=2 to L%
200 for J%=1 to 6 'This loop examines each digit of P
210 if val(mid(P$,I%,1))=A%(J%) then C%=1
220 next:next
230 if C%=0 then inc M% 'If criteria filled count member (m%)
240 wend
250 return

```

Table 4. The first 100 terms in the prime-digital sub sequence.

2	3	5	7	23	37	53	73	223	227
233	257	277	337	353	373	523	557	577	727
733	757	773	2237	2273	2333	2357	2377	2557	2753
2777	3253	3257	3323	3373	3527	3533	3557	3727	3733
5227	5233	5237	5273	5323	5333	5527	5557	5573	5737
7237	7253	7333	7523	7537	7573	7577	7723	7727	7753
7757	22273	22277	22573	22727	22777	23227	23327	23333	23357
23537	23557	23753	23773	25237	25253	25357	25373	25523	25537
25577	25733	27253	27277	27337	27527	27733	27737	27773	32233
32237	32257	32323	32327	32353	32377	32533	32537	32573	33223

Table 5. Comparison of results.

k	1	2	3	4	5	6	7
Computer count:							
m	4	15	38	128	389	1325	4643
log(m)	0.6021	1.1761	1.5798	2.1072	2.5899	3.1222	3.6668
n	13	64	472	3771	30848	261682	2275350
m/n	0.30769	0.23438	0.08051	0.03394	0.01261	0.00506	0.00204
Theoretical estimates:							
m	4	11	34	109	364	1253	4395
log(m)	0.5922	1.0430	1.5278	2.0365	2.5615	3.0980	3.6430
n	7	55	421	3399	28464	244745	2146190
m/n	0.50000	0.20000	0.08000	0.03200	0.01280	0.00512	0.00205

**Theorem:**

The Smarandache prime-digital sub sequence is infinite.

**Proof:**

We recall the prime counting function  $\pi(x)$ . The number of primes  $p \leq x$  is denoted  $\pi(x)$ . For sufficiently large values of  $x$  the order of magnitude of  $\pi(x)$  is given by  $\pi(x) \approx \frac{x}{\log x}$ . Let  $a$

and  $b$  be digits such that  $a > b \neq 0$  and  $n(a, b, k)$  be the approximate number of primes in the interval  $(b \cdot 10^k, a \cdot 10^k)$ . Applying the prime number counting theorem we then have:

$$n(a, b, k) \approx \frac{10^k}{k} \left( \frac{a}{\log 10 + \frac{\log a}{k}} - \frac{b}{\log 10 + \frac{\log b}{k}} \right) \quad (1)$$

Potential candidates for members of the prime-digital sub sequence will have first digits 2, 3, 5 or 7, i.e. for a given  $k$  they will be found in the intervals  $(2 \cdot 10^k, 4 \cdot 10^k)$ ,  $(5 \cdot 10^k, 6 \cdot 10^k)$  and  $(7 \cdot 10^k, 8 \cdot 10^k)$ . The approximate number of primes  $n(k)$  in the interval  $(10^k, 10^{k+1})$  which might be members of the sequence is therefore:

$$n(k) = n(4, 2, k) + n(6, 5, k) + n(8, 7, k) \quad (2)$$

The theoretical estimates of  $n$  in table 5 are calculated using (2) ignoring the fact that results may not be all that good for small values of  $k$ .

We will now find an estimate for the number of candidates  $m(k)$  which qualify as members of the sequence. The final digit of a prime number  $> 5$  can only be 1, 3, 7 or 9. Assuming that these will occur with equal probability only half of the candidates will qualify. The first digit is already fixed by our selection of intervals. For the remaining  $k-1$  digits we have ten possibilities, namely 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 of which only 2, 3, 5 and 7 are good. The probability that all  $k-1$  digits are good is therefore  $(4/10)^{k-1}$ . The probability  $q$  that a candidate qualifies as a member of the sequence is

$$q = \frac{1}{2} \cdot \left( \frac{4}{10} \right)^{k-1} \quad (3)$$

The estimated number of members of the sequence in the interval  $(10^k, 10^{k+1})$  is therefore given by  $m(k) = q \cdot n(k)$ . The estimated values are given in table 5. A comparison between the computer count and the theoretically estimated values shows a very close fit as can be seen from diagram 5 where  $\log_{10} m$  is plotted against  $k$ .

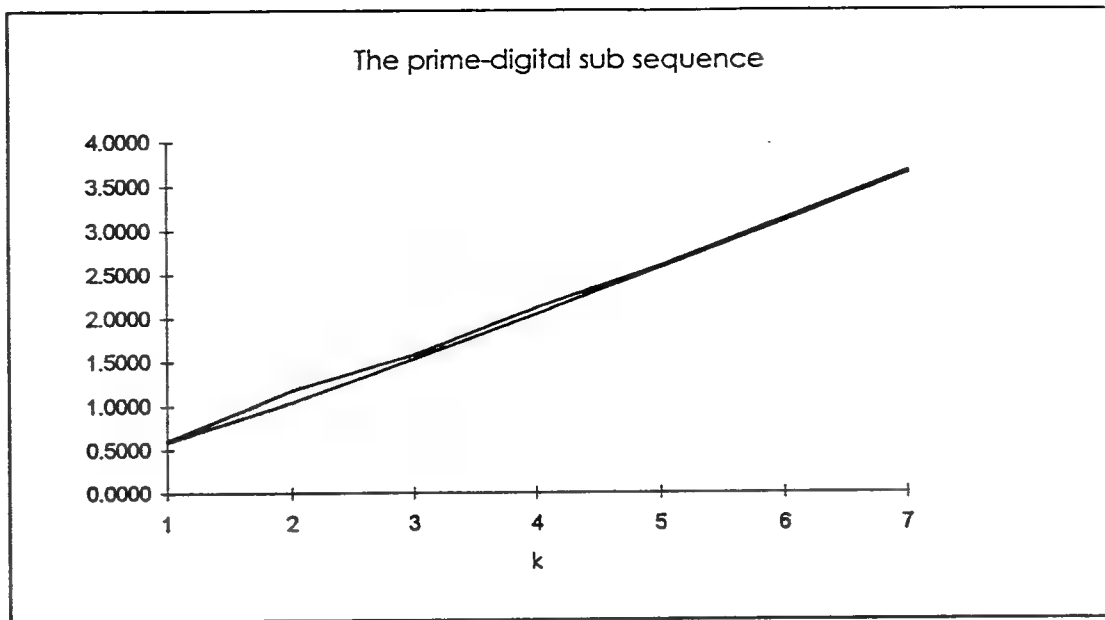


Diagram 5.  $\log_{10} m$  as a function of  $k$ . The upper curve corresponds to the computer count.

For large values of  $k$  we can ignore the terms  $\frac{\log a}{k}$  and  $\frac{\log b}{k}$  in comparison with  $\log 10$  in (1).

For large  $k$  we therefore have

$$n(a, b, k) \approx \frac{(a-b)10^k}{k \log 10} \quad (1')$$

and (2) becomes

$$n(k) \approx \frac{4 \cdot 10^k}{k \log 10} \quad (2')$$

Combining this with (3) we get

$$m(k) \approx \frac{5 \cdot 2^{2k}}{k \log 10} \quad (4)$$

From which we see (apply for instance l'Hospital's rule) that  $m(k) \rightarrow \infty$  as  $k \rightarrow \infty$ . A fortiori the prime-digital sub sequence is infinite.

## V. Smarandache Concatenated Sequences

Smarandache formulated a series of very artificially conceived sequences through concatenation. The sequences studied below are special cases of the Smarandache Concatenated S-sequence.

**Definition:** Let  $G = \{g_1, g_2, \dots, g_k, \dots\}$  be an ordered set of positive integers with a given property  $G$ . The corresponding concatenated S.G sequence is defined through

$$S.G = \{a_i: a_1 = g_1, a_k = a_{k-1} \cdot 10^{1+\log_{10} g_k} + g_k, k \geq 1\}$$

In table 6 the first 20 terms are listed for three cases, which we will deal with in some detail below.

Table 6. The first 20 terms of three concatenated sequences

The S.odd sequence	The S.even sequence	The S.prime sequence
1	2	2
13	24	23
135	246	235
1357	2468	2357
13579	246810	235711
1357911	24681012	23571113
1357911113	2468101214	2357111317
13579111315	246810121416	235711131719
1357911131517	24681012141618	23571113171923
135791113151719	2468101214161820	2357111317192329
13579111315171921	246810121416182022	235711131719232931
1357911131517192123	24681012141618202224	23571113171923293137
135791113151719212325	2468101214161820222426	2357111317192329313741
13579111315171921232527	246810121416182022242628	235711131719232931374143
1357911131517192123252729	24681012141618202224262830	23571113171923293137414347
135791113151719212325272931	2468101214161820222426283032	2357111317192329313741434753
13579111315171921232527293133	246810121416182022242628303234	235711131719232931374143475359
1357911131517192123252729313335	24681012141618202224262830323436	23571113171923293137414347535961
135791113151719212325272931333537	2468101214161820222426283032343638	2357111317192329313741434753596167
13579111315171921232527293133353739	246810121416182022242628303234363840	235711131719232931374143475359616771
1357911131517192123252729313335373941	24681012141618202224262830323436384042	23571113171923293137414347535961677173

**Case 1.** The S.odd sequence is generated by choosing  $G=\{1,3,5,7,9,11,\dots\}$ . Smarandache asks how many terms in this sequence are primes and as is often the case we have no answer. But for this and the other concatenated sequences we can take a look at a fairly large number of terms and see how frequently we find primes or potential primes. As in the case of prime-product sequence we will resort to Fermat's little theorem to find all primes/pseudo-primes among the first 200 terms. If they are not too big we can then proceed to test if they are primes. For the S.odd sequence there are only five cases which all were confirmed to be primes using the elliptic curve prime factorization program. In table 7 # is the term number, L is the number of digits of N and N is a prime number member of the S.odd sequence.:

Table 7. Prime numbers in the S.odd sequence

#	L	N
2	2	13
10	15	135791113151719
16	27	135791113151719212325272931
34	63	135791113151719212325272931333537394143454749515355575961636567
49	93	135791113151719212325272931333537394143454749515355575961636567697173757779818385878991939597

Term #201 is a 548 digit number.

**Case 2.** The S.even sequence is generated by choosing  $G=\{2,4,6,8,10, \dots\}$ . The question here is : How many terms are  $n$ th powers of a positive integer?

A term which is a  $n$ th power must be of the form  $2^n \cdot a$  where  $a$  is an odd  $n$ th power. The first step is therefore to find the highest power of 2 which divides a given member of the sequence, i.e. to determine  $n$  and at the same time we will find  $a$ . We then have to test if  $a$  is a  $n$ th power. The Ubasic program below has been implemented for the first 200 terms of the sequence. No  $n$ th powers were found.

**Ubasic program:** (only the essential part of the program is listed)

```

60 N=2
70 for U%=4 to 400 step 2
80 D%=int(log(U%)/log(10))+1      'Determine length of U%
90 N=N*10^D%+U%                  'Concatenate U%
100 A=N:E%=0
110 repeat
120 A1=A:A=A\2:inc E%              'Determine E% (=n)
130 until res<0

```

```

132 dec E%:A=A1          'Determine A (=a)
140 B=round(A^(1/E%))
150 if B^E%=A then print E%,N    'Check if a is a nth power
160 next
170 end

```

So there is not even a perfect square among the first 200 terms of the S.even sequence. Are there terms in this sequence which are  $2^p$  where  $p$  is a prime (or pseudo prime). With a small change in the program used for the S.odd sequence we can easily find out. Strangely enough not a single term was found to be of the form  $2^p$ .

**Case 3.** The S.prime sequence is generated by  $\{2, 3, 5, 7, 11, \dots\}$ . Again we ask: - How many are primes? - and again we apply the method of finding the number of primes/pseudo primes among the first 200 terms.

There are only 4 cases to consider. Terms #2 and #4 are primes, namely 23 and 2357. The other two cases are: term #128 which is a 355 digit number and term #174 which is a 499 digit number.

#128

```

235711131719232931374143475359616771737983899710110310710911312713113713914915115716316717317918
11911931971992112232272292332392412512572632692712772812832933073113133173313373473493535936737
337938338939740140941942143143343944344945746146346747948749149950350952152354154755756356957157
7587593599601607613617619631641643647653659661673677683691701709719

```

#174

```

235711131719232931374143475359616771737983899710110310710911312713113713914915115716316717317918
11911931971992112232272292332392412512572632692712772812832933073113133173313373473493535936737
337938338939740140941942143143343944344945746146346747948749149950350952152354154755756356957157
758759359960160761361761963164164364765365966167367768369170170971972773373974375175776176977378
779780981182182382782983985385785986387788188388790791191992993794194795396797197798399199710091
0131019102110311033

```

Are these two numbers prime numbers?

# THE SYSTEM - GRAPHICAL ANALYSIS OF SOME NUMERICAL SMARANDACHE SEQUENCES

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The system - graphical analysis results of some numerical Smarandache sequences are adduced. It is demonstrated that they possess of the big aesthetic, cognitive and applied significance.

## 1 Introduction

The analytical investigation<sup>1</sup> of some 6 numerical Smarandache sequences<sup>2</sup> permitted to state that the terms of these sequences are given by the following general recurrent expression

$$a_{\varphi(n)} = \sigma(a_n 10^{\psi(a_n)} + a_n + 1), \quad (1)$$

where  $\varphi(n)$  and  $\psi(a_n)$  — some functions;  $\sigma$  — operator. In this paper we will denote all numerical sequences, yielded by (1), as Smarandache sequences of 1st kind, and analyse ones by system - graphical methods. The main goal of the present research is to demonstrate that the system - graphical analysis results of numerical Smarandache sequences of 1st kind possess of the big aesthetic, cognitive and applied significance.

## 2 System - graphical analysis of some Smarandache sequences of 1st kind

### 1. Smarandache numbers

$$1, 12, 123, 1234, 12345, 123456, \dots \quad (2)$$



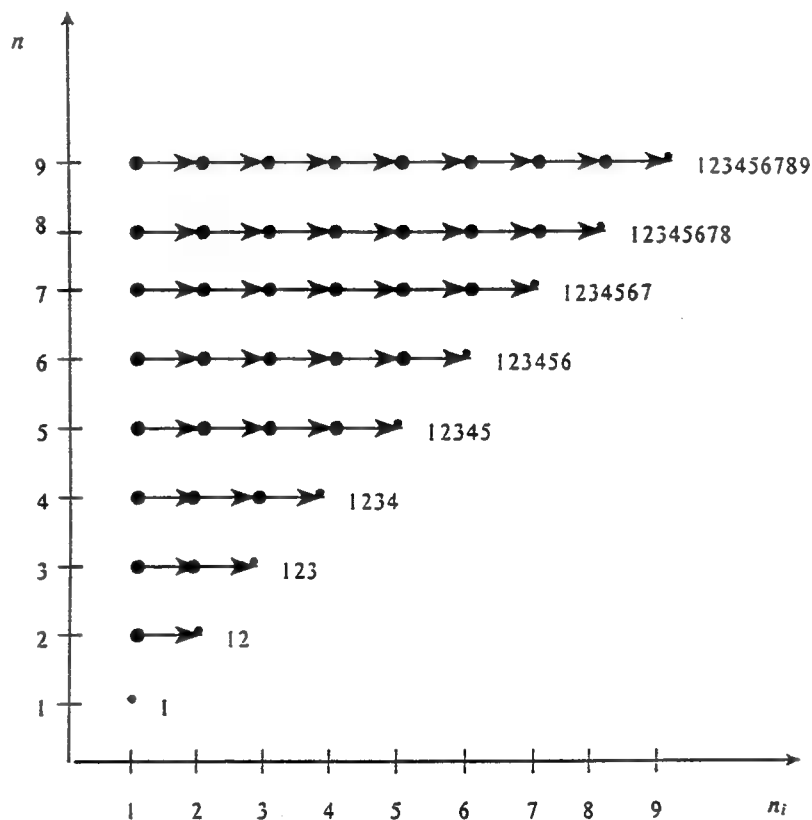


Fig. 1. Graphical image of the first nine terms of  $S_1$ -series.

we shall call *numbers of  $S_1$ -series*. Graphical image of the first nine terms of  $S_1$ -series is given in Fig. 1. For these numbers we introduce<sup>1</sup> an operator  $\Lambda^{-k}$ , making  $k$ -truncation the numbers (2) from the left and/or from the right: for instance, if  $k = 1$  then  $(\Lambda^{-k} 123) = 23$  and  $(123\Lambda^{-k}) = 12$ .

It is evident from Fig. 1 that one may use numbers of  $S_1$ -series with  $\Lambda^{-k}$  operator as a *standardizative representation* of any quantity characteristics of investigated object in such cases when the values of these characteristics are limited from the left and/or the right and uniformly discrete. For instance, one may use mentioned standardization in visual control device of sound level in

audio-techniques, in the information decoding and transmitting systems and so on.

## 2. Smarandache numbers

$$1, 11, 121, 1221, 12321, 123321, 1234321, \dots \quad (3)$$

we shall call *numbers of  $S_2$ -series*. The numbers (3) possess the mirror-symmetric properties evinced in graphical image of ones (see Fig. 2) by the presence of reverse motion arcs. It is easy to find that there are a great number of technical and physical objects, using the same principle of action as one showed in Fig. 2.

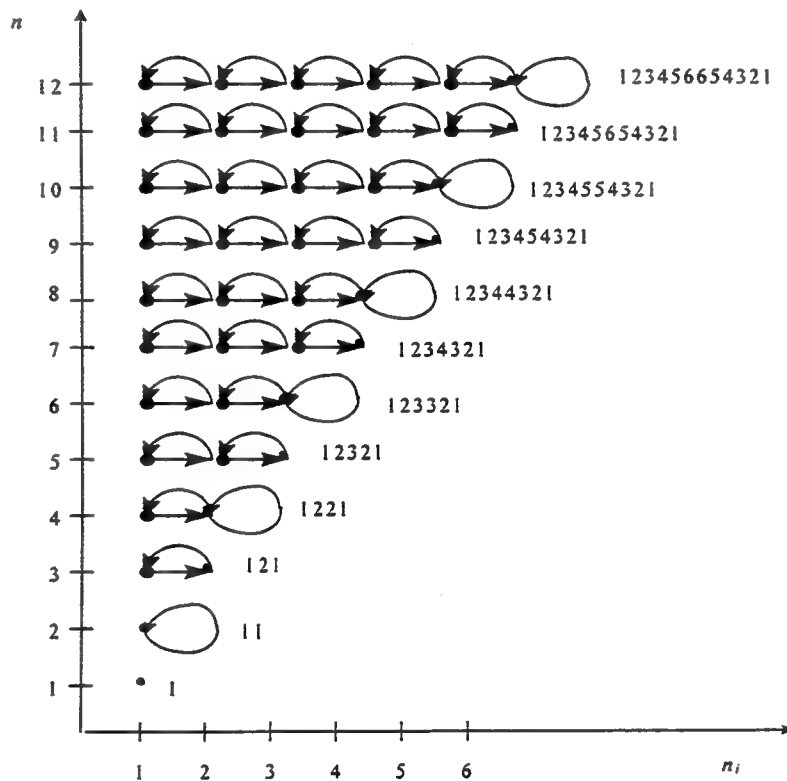


Fig. 2. Graphical image of the first twelve terms of  $S_2$ -series.

In particular, ■ standardizative representation in terms of Smarandache numbers of  $S_2$ -series can be made for reverse connection circuits in the different control and handling systems; for the suitable graphical representation of any systems in which for complete description of system state the knowledge of  $n$  last states is required; for coding information on effects of "staying waves" and so on.

### 3. Smarandache numbers

$$1, 212, 32123, 4321234, 543212345, 65432123456, \dots \quad (4)$$

we shall call *numbers of  $S_3$ -series*. By analysing graphical image of  $S_3$ -series terms given in Fig. 3 one can conclude this image is similar to that given in Fig. 2. Indeed, figures of  $S_3$ -series terms differ from the ones of  $S_2$ -series terms only by reverse orientation in space (a suitable interpretation for description reversible physical phenomenon) or by another initial state (— for the theory of automates).

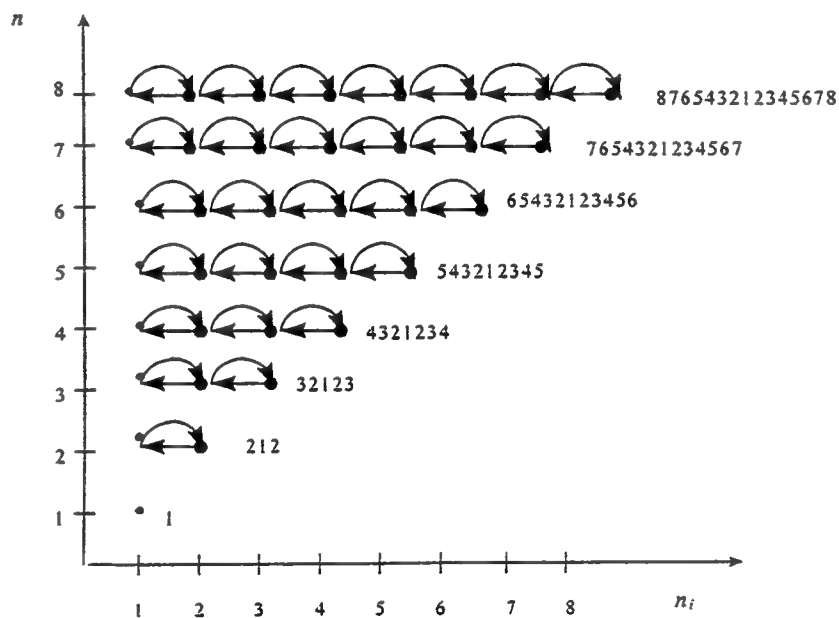


Fig. 3. Graphical image of the first eight terms of  $S_3$ -series.

Thus, the numbers of  $S_3$ -series can get the same applications in standardizative representations of quantity characteristics as the numbers of  $S_2$ -series, though they are less useful because its graphic image structure more poor than one that numbers of  $S_2$ -series have.

#### 4. Smarandache numbers

$$1, 23, 456, 7891, 23456, 789123, 4567891, \dots \quad (5)$$

we shall call *numbers of  $S_4$ -series*. It's graphical image is given in Fig. 4. In distinction from the terms of considered Smarandache series the ones of  $S_4$ -series consist of only numbers from 1 to 9. Thus, after 17th term of  $S_4$ -series 23456789123456789 the successive ones do not enrich  $S_4$ -series since any sequence

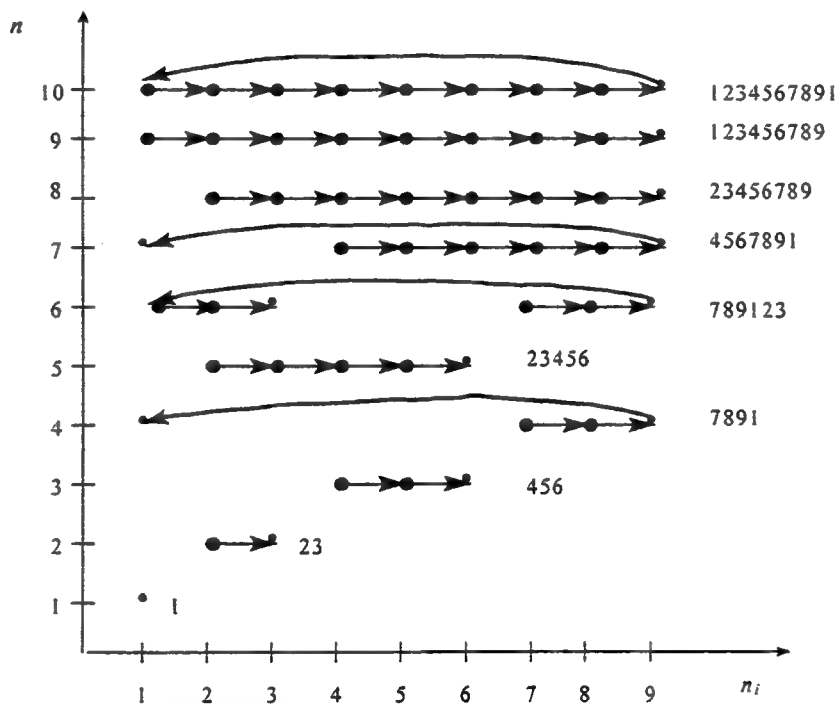


Fig. 4. Graphical image of the first ten terms of  $S_4$ -series.

from 9 or less different successive digits can be obtained from 17th term of  $S_4$ -series by truncation from the left and right  $\Lambda^{-k}$  operators. However, in spite of mentioned lack the standardizative representation of quantity characteristics of local computer nets by terms of  $S_4$ -series may be quite useful. In particular, such Smarandache numbers can reflect the principle of transmitting data packets from one local station to another. Besides one may use standardization by  $S_4$ -terms for description of recurrent relations between sequence elements or some processes, described usually by Markov's chains.

#### 5. Smarandache numbers

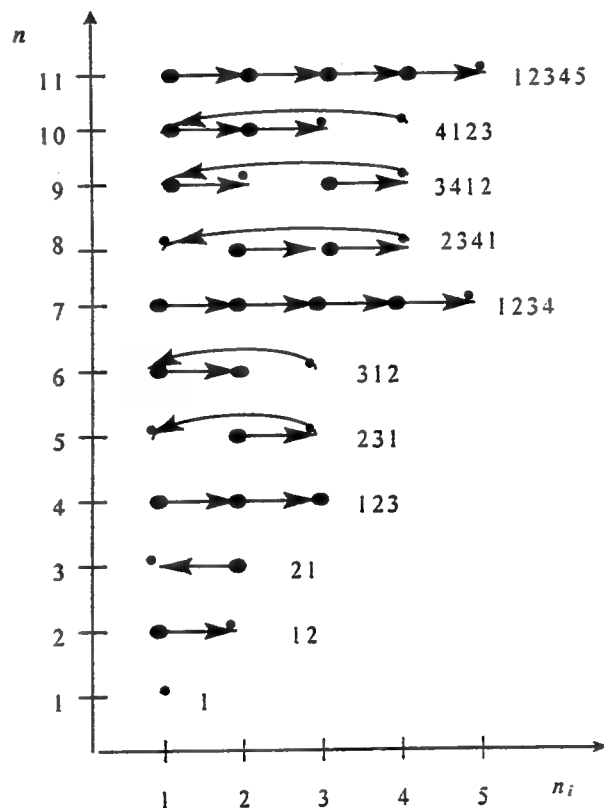


Fig. 5. Graphical image of the first eleven terms of  $S_5$ -series.

$$1, 12, 21, 123, 231, 312, 1234, 2341, 3412, 4123, 12345, \dots \quad (6)$$

we shall call *numbers of  $S_5$ -series*. It is evident that  $S_5$ -series contains all the numbers of  $S_1$ -series and some additional numbers. Successive-circular properties of this series are well-shaped in Fig. 5. The analysis of the  $S_5$ -series graphical image permits to find applications fields where  $S_5$ -series terms as standardizative representation of object characteristics can be used: these are fields where some look over several states of objects is required. For instance, technical diagnostics, systems of processes handling, theory of selecting and taking the decision may be named.

#### 6. Smarandache numbers

$$12, 1342, 135642, 13578642, 13579108642, \dots \quad (7)$$

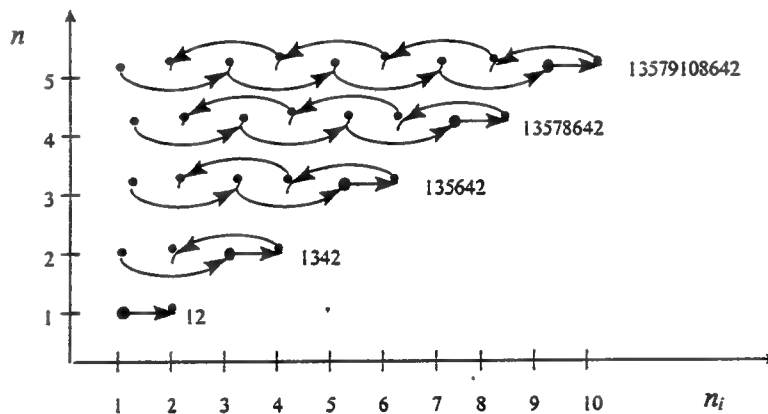


Fig. 6. Graphical image of the first five terms of  $S_6$ -series.

we shall call *numbers of  $S_6$ -series*. These numbers also as numbers (6) have circular properties with the uniform structure. Graphical image of (7) is given in Fig. 6. By analysing Fig. 6 one may note that presented images may describe the test procedure rounds the all elements of system with minimal steps and aim to finish the round in the element the nearest to the initial one. Besides it turns out

that a term of (7) being divided into two subterms can serve as standardizative representation of two simultaneous processes. In particular, such standardization of  $S_6$ -series terms can be applied for parallel signal processing or parallel design processes.

### 3 System - graphical analysis of numbers of $S_2$ - and $S_3$ -series

Divide! a set of  $S_2$ -series numbers (3) into two different subsequences:

1)  $a_1=1, a_2=121, a_3=12321, a_4=1234321, \dots$

2)  $b_1=11, b_2=1221, b_3=123321, b_4=12344321, \dots$

The numbers of the first and the second subsequences we shall call *A*- and *B*-numbers correspondingly.

In Sect. 2 Smarandache numbers were presented in the proper constructions on the number axes. It is naturally to expect that employment not only number axes but the whole plane and different geometrical figures for representation Smarandache numbers will permit to reveal new interesting properties of ones, explain by ones a great number of technical and nature processes, study more deeply its peculiarities and preferences. In this section we consider the only mentioned above *A*- and *B*-numbers and also  $S_3$ -series numbers (4).

Firstly we explain how to construct numerical circumferences from Smarandache numbers:

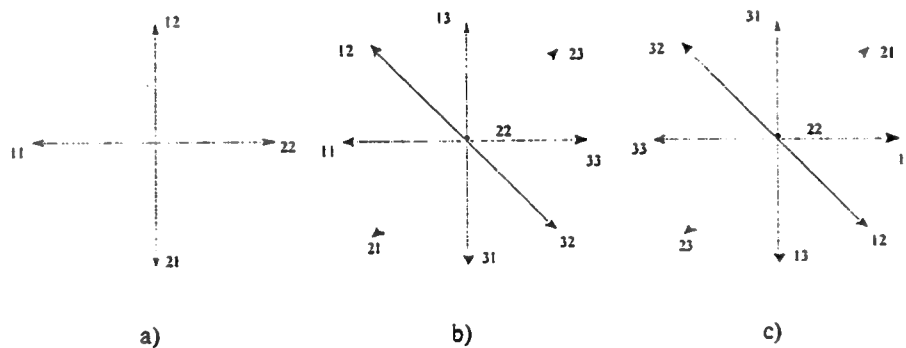


Fig. 7. The graphical images of the second (a) and the third terms of *A*- and *B*-subsequences and the third term (c) of  $S_3$ -series.

a) the first terms of  $A$ - and  $B$ -subsequences and of  $S_3$ -series convert into the point;

b) the graphical images of the second and the third terms of these sequences are given in Fig. 7 correspondingly.

By analysing Fig. 7 one can easily find the graphical forms of the representation for next terms of these sequences. Namely, to construct the graphical image of  $n$ -th term of mentioned above Smarandache sequences the following algorithm may be used:

1. To draw the circumference and two perpendicular lines crossed in the centre of circle.

2. To denote the tops of four rays going from the point of the cross as  $1l$ ,  $1n$ ,  $nn$ ,  $nl$  consequently in the forward of clock hand.

3. To divide every sector into  $n-1$  equal parts by drawing additional rays with proper mark.

At such representation of Smarandache sequences terms these ones produce the subsequences. For example,

a) the second term of  $A$ - or  $B$ -subsequences produce the proper  $2 \times 2$  series of subsequence

$$11, 22, 22, 11; 12, 21, 21, 12; \quad (8)$$

b) the third term of  $A$ - or  $B$ -subsequences produce the proper  $3 \times 3$  series of subsequence

$$\begin{aligned} 11, 22, 33, 33, 22, 11; 12, 22, 32, 32, 22, 12; \\ 13, 22, 31, 31, 22, 13; 21, 22, 23, 23, 22, 21; \end{aligned} \quad (9)$$

and so on.

Among splendid peculiarities of graphical images, depicted in Fig. 7, we point that the all Smarandache circumferences reveal Magic properties: they have a constant sum for the elements located in the diameters of the circumference. It is very interesting to confront the ancient Chinese hexagrams, located on the circumference (see Fig. 8a), with numbers of subsequence (9), by which the tops of diameters are marked (Fig. 8b).

By deleting commas in (8) and (9) and combining two-digit elements in single terms one obtains extended representation of Smarandache sequences terms.



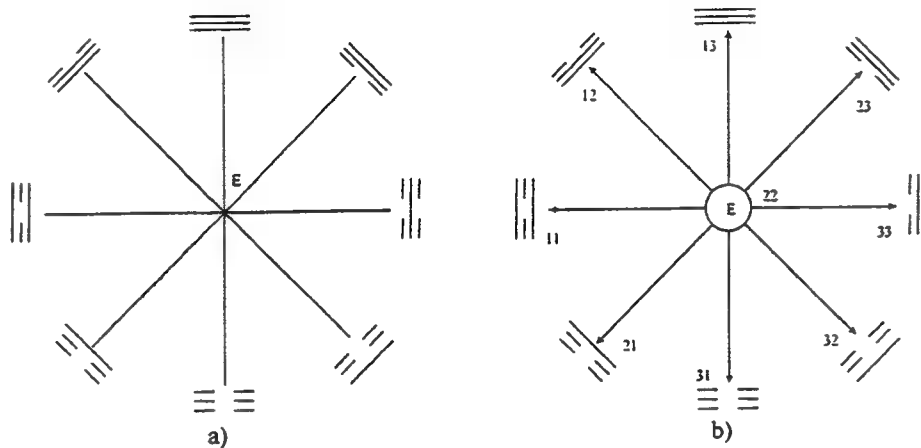


Fig. 8. The graphical confrontation of the ancient Chinese hexagrams with numbers of subsequence (9).

Thus, graphical images of Smarandache sequences terms, used as standardizative representations of the objects allow both decomposition of the object representation (analysis of the object) and combination of the object representation (synthesis of a new object). This graphical technique is similar to operators of truncation  $\Lambda^{-k}$  and extending  $\Lambda^{+k}$  of series terms<sup>1</sup>, but more flexible.

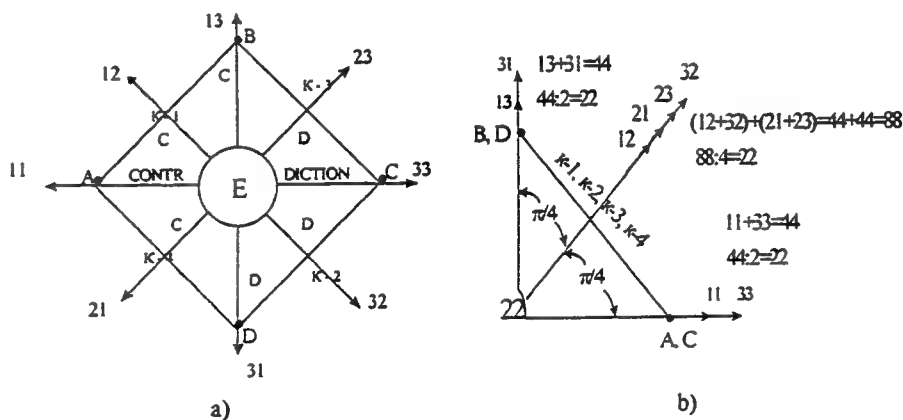


Fig. 9. The Aristotelian logical (a) and the reduced logical (b) squares.

Curtailment of these graphical images is seemed to be very interesting since it reflects the ways of simplification of real instruments and devices. For example, if one takes into account that Aristotelian logical square can be superposed into the graphical representation of *A*- or *B*-subsequences third term (see Fig. 9a) then the circumference may be reduced one fourth of the circle (Fig. 9b).

Very curious graphical images with Smarandache circumferences are revealed when one draws the track of point *M* and *M'* movement along the circumference, and this track will be shown not on the whole diagram, but on the reduced one to  $1/8$  of the circle (see Fig. 10).

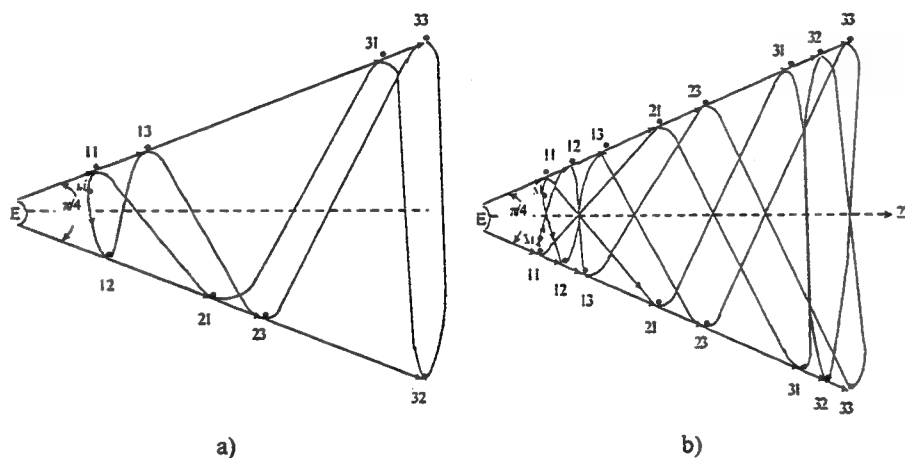


Fig. 10. The track of points *M* (a) and both *M* and *M'* (b) movement along the circumference.

Very important for understanding internal properties of Smarandache numbers are images given in Fig. 11(a, b), where the third term of *A*- or *B*-subsequences is depicted. Indeed, it shows the quantitative characteristics of Smarandache numbers even in such case when numerical information, adduced in Fig. 11(a, b), is absent. We pay attention, that in Fig. 11b circumference with unit diameter and the graphical quantitative characteristic of Smarandache number are depicted simultaneously.



subsequences contains in the implicit form picture of an aeroplane. We assume that by using discussed system graphical analysis methods one may reveal some another unexpected graphical information, contained in some Smarandache numbers.

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**THE FIRST INTERNATIONAL CONFERENCE  
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**Objectives:** To bring together mathematicians, professors, teachers, students, etc. interested in Smarandache type functions, sequences, algorithms, operations, criteria, theorems, etc., but contributed papers and/or plenary lectures were invited from all areas of Number Theory.

**Papers:** Contributions (either 10 or 30 minutes talks, as well as plenary lectures) on Number Theory and especially on any of the Smarandache type notions are now published in the Proceedings of the Conference.

**Invited Speakers:** I. Balacenoiu, M. Bencze, I. & S. Cojocaru, N. Ivaschescu, G. Policarp, M. & P. Popescu, E. Burton, C. Rocsoreanu, E. & N. Radescu, St. Smarandoiu, St. Zafir, L. Tutescu, H. Ibstedt (Sweden), J. Normand (France), M. Mudge (England), P. Gronas (Norway), C. Ashbacher (USA), K. Tauscher (Australia), J. Rodriguez (Mexico), K. Atanasov (Bulgaria), Fujii Akio (Japan), V. Suceveanu (R. Moldova), F. Luca (Syracuse Univ., NY), Y. Chebrakov (Russia), S.M. Ruiz (Spain), S. Porubsky (Czech R.), etc.

**Sponsors:** Erhus Univ. Press (Vail, AZ) & Number Theory Association (Craiova, Romania).

**Organizers:** C. Dumitrescu & V. Seleacu, Univ. of Craiova. This Conference was organized under the auspices of UNESCO.

**Paper and Abstract deadline:** was July 31, 1997.

**No registration fee.**

**Free trip in the Craiova's neighborhood.**

**Information:** Dr. C. Dumitrescu, Mathematics Department, University of Craiova, R-1100 Romania, tel.: (40) 51-125302, fax: (40) 51-413728 [for Dumitrescu], e-mail: florenta@oltenia.ro.

**Local expenses (housing, meal) were provided by the organizers.**

Dr. C. Dumitrescu, Chairman of the Organizing Committee